SUBJECT: ANALYSIS IV

LESSON: CONNECTEDNESS IN METRIC SPACES

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1. Learning outcomes

In this lesson, we have introduced the notion of connectedness and proved some of its equivalent forms. We will discuss the behavior of connectedness under continuous functions and show that connectedness is a topological property. We shall also see the behavior of connected subspaces with respect to union and intersection. Then we discuss the concept of connected components and work out their properties. After that we study connectedness with respect to the metric space $(\mathbb{R}, d)$. We will show that $(\mathbb{R}, d)$ is connected and a subset of $\mathbb{R}$ is connected if and only if it is interval. In this lesson, we also study the behavior of connectedness with respect to product metric spaces. Lastly, we shall study two variants of connectedness, namely, the total connectedness and path wise connectedness.

2. Introduction

Continuous functions play a central role in defining and characterizing many concepts in metric spaces. Connectedness is one of such important concepts. The philosophy of connectedness is to keep the whole space intact into a single piece. It does not let the space divide into pieces which lie apart from each other. This is in fact the intrinsic property of a connected space. Thus the idea of connectedness is very simple and intuitive. In physical sense, a space is called connected if it cannot be split in two or more pieces that never touch or two pieces not having a common boundary. Another way to look at connectedness is through boundary. Clearly, boundary of any object links it to outside world. Thus if there is any object in the space not having any boundary, then it gets disconnected from rest of the space. We will see that the concept of connectedness in mathematics corresponds with this intuitive idea.

We will see in this chapter that a singleton subset of any metric space is always connected. So a natural question arises, in a disconnected metric space, do there exist other connected subspaces too. The answer is yes in some special cases. We will also discuss the maximal connected subspaces of a given metric space, which we term as the connected components. We then study the properties enjoyed by these connected components. Further, we will discuss $(\mathbb{R}, d)$ and show that $(\mathbb{R}, d)$ is connected and a subset of $\mathbb{R}$ is connected if and only if it is an interval. Later, we study the behavior of connectedness with respect to product metric spaces. We culminate this by introducing two variants of connectedness, namely, the total connectedness and path-wise connectedness.

2. Connectedness

NOTATIONS

Notations	**Meaning**
∂A	boundary of the set A
A^0	Interior of A
$\bar{A}$	closure of A
$(\mathbb{R}, d)$	metric space of real numbers equipped with the usual metric
$dist(x, A)$	$inf\{d(x,a) : a \in A\}$
A^c or $X \sim A$	Complement of A

Definition 2.1: Let (X, d) be a metric space and let A be a subset of X. Then an element $x \in X$ is said to be a **boundary point** of A if and only if

$$dist(x, A) = 0 = dist(x, A^C).$$

The set of all boundary points of A denoted by ∂A is called the boundary of A.

Definition 2.2: A subset A of a metric space (X, d) is said to be **open** if $A \cap \partial A = \phi$ i.e., it does not intersect it's interior.

Definition 2.3: A subset A of a metric space (X, d) is said to be **closed** if it contains it's interior i.e., $\partial A \subseteq A$.

Definition 2.4: A metric space (X, d) is said to be **connected** if X cannot be expressed as a union of two disjoint non – empty open subsets of X i.e., if $X = A \cup B$ where A and B are disjoint open subsets of X, then either $A = \phi$ or $B = \phi$. A metric space which is not connected is said to be **disconnected**.

Example 2.1 If (X, d) is a discrete metric space containing more than one element, then it is disconnected. Since in a discrete metric space every subset of X is open, therefore for any $x \in X$, the sets $\{x\}$ and $X - \{x\}$ are non-empty disjoint open subsets of X. Now as $X = \{x\} \cup (X - \{x\})$, thus X is expressed as a union of two disjoint open subsets of X. Hence (X, d) is disconnected.

Criteria for Connectedness

Theorem 2.1: For a metric space (X, d) the following are equivalent:

(i). Every proper subset of X has non – empty boundary in X. (***Boundary Criterion***)

(ii). No proper subset of X is both open and closed. (***Open and Closed Criterion***)

(iii). X is not the union of two disjoint non – empty open subsets of X. (***Open Union Criterion***)

(iv). X is not the union of two disjoint non – empty closed subsets of X. (***Closed Union Criterion***)

(v). Either $X = \phi$ or the only continuous functions from X to the discrete metric space $\{0, 1\}$ is constant. (***Continuity Criterion***)

Proof: We prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv)$\Rightarrow$ (v)$\Rightarrow$ (i)

(i) $\Rightarrow$ (ii) Let for each proper subset A of X, $\partial A \neq \phi$, Let if possible, there exists a proper subset A of X which is both open and closed. Since A is closed, therefore $\partial A \subseteq A$ i.e., $\partial A \cap A = \partial A$. Also, as A is open, $\partial A \cap A = \emptyset$ which implies that $\partial A = \emptyset$, a contradiction.

(ii) $\Rightarrow$ (iii) Suppose there does not exist any proper subset of X which is both open and closed. Let if possible X be expressible as the union of two proper disjoint open subsets G and H of X, i.e., $X = G \cup H$, $G \cap H = \phi$. Now since $G = X - H$ and H is open, therefore G is closed. Thus G is a proper subset of X which is both open and closed. This contradicts the hypothesis.

(iii) $\Rightarrow$ (iv) Let X be not expressible as the union of two proper disjoint open subsets of X. We show that X cannot be expressed as union of two proper disjoint closed subsets of X. Suppose the contrary, F_1 and F_2 be two proper closed subsets of X such that $X = F_1 \cup F_2$ and $F_1 \cap F_2 = \phi$. Then $F_2 = X - F_1$ and $F_1 = X - F_2$. Now since F_1 and F_2 are closed subsets of X, therefore both F_1 and F_2 are open subsets of X (being complements of closed subsets). Thus X is expressible as the union of two proper disjoint open subsets F_1 and F_2 of X, contradicting the hypothesis.

(iv) $\Rightarrow$ (v) Let us suppose that X be not the union of two disjoint non – empty closed subsets of X. If $X = \emptyset$, then we have nothing to prove. Thus let $X \neq \emptyset$ and there exists a non–constant continuous function f from X onto the two element $\{0,1\}$ discrete metric space. Since f is non – constant, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are non-empty subsets of X. Further since $\{0\}$ and $\{1\}$ are closed subsets of discrete metric space and since f is continuous $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are closed subsets of X. Then clearly, $X = f^{-1}(\{0\}) \cup f^{-1}(\{1\})$and $f^{-1}(\{1\}) \cap f^{-1}(\{1\}) = \phi$. Hence X is expressed as the union of two disjoint non-empty closed subsets of X, which contradicts the given hypothesis.

(v) $\Rightarrow$ (i) Suppose that for any metric space (X,d) either $X = \phi$ or the only continuous functions from X to the discrete metric space $\{0,1\}$ is constant. If $X = \emptyset$, then (i) holds vacuously. Thus let $X \neq \emptyset$. We show that every proper subset of X has non-empty boundary. Suppose on the contrary, there exists a proper subset A of X such that $\partial A = \phi$. Now since $\partial A = \partial(A^c) = \emptyset$, therefore both A and A^c are open subsets of X.

Define a function $f : X \rightarrow \{0,1\}$ where $\{0,1\}$ is endowed with discrete metric as

$$f(x) = \begin{cases} 0 & if \ x \in A \\ 1 & if \ x \notin A \end{cases}$$

Since $\{0,1\}$ is a discrete metric space, its open sets are ϕ , $\{0\}$, $\{1\}$, and $\{0,1\}$. It is easy to see that $f^{-1}(\phi) = \phi$, $f^{-1}(\{0\}) = A$, $f^{-1}(\{1\}) = A^C$ and $f^{-1}(\{0,1\}) = X$. Now since each of ϕ, A , A^C, and X are open in X, therefore we see that the inverse image of each open subset of the discrete metric space is open in X. Thus f is a continuous function. Then by the hypothesis, f is a constant function.

Therefore either $A = f^{-1}(\{0\}) = \emptyset$ or $A^c = f^{-1}(\{1\}) = \emptyset$. Thus either $A = \emptyset$ or $A = X$, contradiction to the fact that A is a proper subset of X. ■

Remark 2.1: From the above theorem it is clear that all the above statements are equivalent to connectedness of a metric space. So, any of these statements can be taken as a definition of connectedness.

However, since boundary condition is close to the intuitive idea of the connectedness it is worth mentioning it separately.

Definition 2.5: A metric space (X,d) is said to be **connected** if the boundary of every proper subset of X is non-empty. Equivalently, a metric space (X,d) is disconnected if there exists a proper subset A of X such that $\partial A = \varphi$.

Example 2.2: Consider the space (S, d_s), where $S = (0,1) \cup (7,8)$ and d_s is the usual distance metric on S. Then both subsets $(0,1)$ and $(7,8)$ of S have empty boundary since all the points of the set $(7,8)$ are at a distance at least 6 from its complement $(0,1)$ and vice versa. Hence the space (S,d_s) is disconnected.

Theorem 2.2: A metric space (X,d) is disconnected if and only if there exist two non – empty subsets A and B of X such that $X = A \cup B$, $A \cap \overline{B} = \varphi = \overline{A} \cap B$.

Proof: Since $A \cap B \subseteq A \cap \bar{B}$ and since $\cap \bar{B} = \phi$, we have $A \cap B = \phi$. Now, let $G = X - \bar{B}$ and $H = X - \bar{A}$. Then G and H are non-empty open subsets of X.
Consider,

$$A \cap \bar{B} = \phi \Rightarrow A \subseteq X - \bar{B} \text{ and}$$
$$\bar{A} \cap B = \phi \Rightarrow B \subseteq X - \bar{A}$$

Thus it follows that

$$\begin{aligned} X &= A \cup B \\ &\subseteq (X - \bar{B}) \cup (X - \bar{A}) \\ &= G \cup H \subseteq X. \end{aligned}$$

Hence $X = G \cup H$. Further,

$$\begin{aligned} G \cap H &= (X - \bar{B}) \cap (X - \bar{A}) \\ &= X - (\bar{A} \cup \bar{B}) \\ &\subseteq X - (A \cup B) = \emptyset. \end{aligned}$$

Thus, X is expressible as a disjoint union of two non-empty open subsets of X and hence it follows that (X,d) is disconnected.

Conversely, Suppose (X,d) is disconnected. Then there exists a pair of disjoint non – empty open subsets A and B of X such that $X = A \cup B$, and $A \cap B = \phi$. We claim that $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$. Let $x \in A$ be a limit point of B. Then each open neighborhood of x in particular A (since A is open) is such that $A \cap B - \{x\} \neq \phi$. But it contradicts the fact that $A \cap B = \phi$. Hence, no point of A can be a limit point of B. Similarly, we can show that no point of B can be a limit of A. Therefore $\bar{A} \cap B = \phi = A \cap \bar{B}$ proving the assertion. ■

Definition 2.6: Two non-empty subsets A and B of a metric space (X,d) are said to be **separated** if $\bar{A} \cap B = \phi = A \cap \bar{B}$. We say that subsets A and B is a **separation of X**.

Remark 2.2: From Definition 2.6 and Theorem 2.2, it is clear that a metric space (X,d) is disconnected if and only if X can be expressed as a union of two separated sets.

Theorem 2.3: Let (X,d) be a disconnected metric space and let A and B be a separation of X, i.e., $X = A \cup B$, $A \cap \overline{B} = \varphi = \overline{A} \cap B$. If $E \subseteq X$ is any connected subset of X, then $E \subseteq A$ or $E \subseteq B$.

Proof: Suppose if possible, $E \cap A \neq \phi$ and $E \cap B \neq \phi$. Let $P = E \cap A$ and $Q = E \cap B$. We claim that P and Q forms the separation of E. Consider,

$$\begin{aligned} E &= E \cap X \\ &= E \cap (A \cup B) \\ &= (E \cap A) \cup (E \cap B) \\ &= P \cup B \end{aligned}$$

Now,

$$P \cap \bar{Q} = (E \cap A) \cap \overline{(E \cap B)}$$

$$= (E \cap A) \cap (\bar{E} \cap \bar{B})$$
$$= (E \cap \bar{E}) \cap (A \cap \bar{B})$$
$$= E \cap \phi = \phi$$

Similarly, we can show that $\bar{P} \cap Q = \phi$. Thus it follows that P and Q forms the separation of E. But this shows that E is disconnected which contradicts the given hypothesis. Hence our assumption is wrong. Thus either $E \cap A = \phi$ and $E \cap B = \phi$ which in turn implies that either $E \subseteq A$ or $E \subseteq B$. ■

3. Connected subsets

Definition 3.1: A subset $S \subseteq X$ of a metric space (X, d) is called a **connected subset** of X if the subspace (S, d_S) of (X, d) is connected metric space.

Example 3.1: Let (X, d) be a metric space. Then empty set and singleton subsets of X are connected subset of X.

Example 3.2: Let $(\mathbb{R}, d)$ be the metric space of real numbers equipped with usual metric. Then φ and all interval subsets of $\mathbb{R}$ (including $\mathbb{R}$) are connected subsets of $\mathbb{R}$.

Example 3.3: Consider $X = \mathbb{R} - \{0\}$ with the metric obtained by restricting the usual metric of $\mathbb{R}$. Consider the set $E = (0, \infty)$. This is clearly a non-empty open subset of X. In fact, we claim that it is both open and closed inside X. To see that E is closed in X, observe that $E^c = X \cap (-\infty, 0)$ is an open subset of X. Thus E is both open and closed subset of X. Hence, in view of Theorem 2.1 (ii) X is disconnected.

Remark 3.1: From Example 3.3, we observe that connectedness is not a hereditary property i.e., a subset of a connected metric space need not always be connected. Further, a subset of a metric space may be connected without the space being connected. To see that a subset of a disconnected space is connected, observe that singleton subsets of any metric space is connected and hence in particular of disconnected metric space.

Theorem 3.1: A metric space (X, d) is connected if and only if each pair of distinct points of X is contained in a connected subset of X.

Proof: First let us suppose that (X, d) is connected and let $x, y \in X$ be such that $x \neq y$. Then $\{x, y\}$ is contained in a connected set namely X which is contained in X itself.

Conversely, let us assume that for each pair of distinct elements of X there exists a connected subset of X containing both x and y. We show that (X, d) is connected. Let us suppose the contrary, that is, (X, d) be disconnected. Then there exist separated subsets A and B of X such that $X = A \cup B$. Since $A \cap B = \phi$, let $x \in A, y \in B$, then $x \neq y$.
Then by the given hypothesis, there exists a connected set $K \subseteq X$ such that $\{x, y\} \subseteq K$. Further, since K is a connected subset of $X = A \cup B$, by Theorem 2.3, we have $K \subseteq A$ or $K \subseteq B$ which implies that $\{x, y\} \subseteq A$ or $\{x, y\} \subseteq B$, a contradiction. This contradiction shows that (X, d) is connected. ■

Alternate Proof:

Let us first suppose that for each pair of distinct elements x and y of X there exists a connected subset E of X containing both x and y. We show that (X , d) is connected. We prove this by contra positive argument. Let (X , d) be disconnected. Then there exists a non-constant continuous function f from (X , d) onto (Y , d*), where $Y = \{0,1\}$. Since, f is non constant, there exists a pair of elements x and y in X such that $f(x) \neq f(y)$. Since x and y are distinct elements of X, by hypothesis, there exists a connected subset E of X such that both $x, y \in E$. Now since restriction of a continuous function defined on X is continuous, therefore, $f|_E$ is a continuous function from E to (Y, d^*). But since $x, y \in E$ and E is connected, therefore $f|_E$ is constant. Therefore, $f|_E(x) = f|_E(y)$ which implies that $f(x) = f(y)$ which contradicts the fact that $f(x) \neq f(y)$.

Conversely, let (X, d) be connected then obviously every pair of distinct elements x, y of X belongs to a connected subset of X, namely X itself. ■

Theorem 3.2: Suppose (X, d) is a metric space and let S is a connected subset of X. Let A be any subset of X such that $S \subseteq A \subseteq \bar{S}$. Then A is connected

Proof: If $S = \phi$ then the result is trivially true as in this case $A = \phi$ which is a connected subset of X. So let us assume that $S \neq \phi$. In order to show that A is connected we show that each continuous function from A onto (Y, d^*) where d^* is the discrete metric on $Y = \{0, 1\}$, is constant. Let $f: A \to (Y, d^*)$ be any continuous function. Since restriction of a continuous function to a subspace is continuous, therefore , $f|_S : S \to (Y, d^*)$ is continuous where $f|_S$ denotes the restriction of f to the subspace S. Further, since S is given to be connected $f|_S$ is constant. Now for each $z \in A$, we have $z \in \overline{S}$ and so there exists a sequence $\{x_n\}$ in S that converges to z. Since f is continuous, the sequence $\{f(x_n)\}$ converges to $f(z)$. But since f is constant on S, we have that $\{f(x_n)\}$ is a constant sequence, so that f has the same value at z as it has on S. As z is arbitrary element of A, we see that f is constant on A. This proves the assertion.

Alternate Proof: Consider any subset A of X such that $S \subseteq A \subseteq \bar{S}$ and A is disconnected. Then there exist two disjoint non-empty open subsets P and Q in (A, d_A) such that $A = P \cup Q$. Since P and Q are open in (A, d_A), there exist open subsets P^* and Q^* of X such that

$$P = P^* \cap A \quad and \quad Q = Q^* \cap A.$$

Thus $A = (P^* \cap A) \cup (Q^* \cap A)$ and since $S \subseteq A$, therefore $S = (P^* \cap S) \cup (Q^* \cap S)$. Now as $A \subseteq \bar{S}$,
$P^* \cap A \neq \emptyset \ and \ Q^* \cap A \neq \emptyset$

$$\Longrightarrow \quad P^* \cap \bar{S} \neq \emptyset \ and \ Q^* \cap \bar{S} \neq \emptyset$$

We claim that $P^* \cap S \neq \varphi$. Since $P^* \cap \bar{S} \neq \varphi$, let $x \in P^* \cap \bar{S}$. This implies $x \in \bar{S}$, i.e., x is a limit point of S and so every neighbourhood of x intersects S, in particular P* being open is a neighbourhood of x, therefore $P^* \cap S \neq \emptyset . Similarly \ \ Q^* \cap S \neq \emptyset$. Hence we have expressed S as a disjoint union of non-empty open subsets $P^* \cap S$ and $Q^* \cap S$ in $(S\, d_S)$. It follows that S is disconnected, a contradiction. Hence our assumption is wrong and A is connected. ■

Corollary 3.2: If S is a connected subset of a metric space, then $\bar{S}$ is connected.

Proof: The result follows from Theorem 3.6, since $S \subseteq \overline{S} \subseteq \overline{S}$.

Problem 3.1: Is it true that if the closure of a set S in a metric space (X, d) is connected, then S itself is connected.

Solution: The answer is no. Let $(\mathbb{R}, d)$ be a metric space of real numbers equipped with the usual metric. Let $S = (0,2) \cup (2,3)$. Then Since $(0,2)$ and $(2,3)$ form a separation of S and so is disconnected. On the other hand, we have $\bar{S} = \overline{(0,2) \cup (2,3)} = [0,2] \cup [2,3]$. Since $[0,2]$ and $[2,3]$ are connected subsets of $\mathbb{R}$ and since $[0,2] \cap [2,3] = \{2\} \neq \phi$, therefore $\bar{S} = [0,2] \cup [2,3]$ is connected.

4. Continuous functions and Connectedness

Connectedness is well behaved with respect to continuous functions.

Theorem 4.1: Let $f : (X, d) \to (X^*, d^*)$ be a continuous function from a connected metric space (X, d) onto a metric space (X^*, d^*). Then (X^*, d^*) is connected.

(In other words, continuous image of a connected metric space is connected.)

Proof: Since f is onto, we have $f(X) = X^*$. In order to show that $f(X)$ is connected, we show that every continuous function from $f(X)$ onto a two elementic discrete metric space (Y, ρ) is constant. Let $g : f(X)\big(= (X^*, d^*)\big) \to (Y, \rho)$ be a continuous function. Since composition of two continuous functions is a continuous function, therefore, $gof : (X, d) \to (Y, \rho)$ is a continuous function. Since (X, d) is connected, gof is constant which implies that g is constant on $f(X)$. Hence (X^*, d^*) is connected. ■

Let us recall that a function $f : (X, d) \to (X^*, \rho)$ is said to be a **Homeomorphism** if

(i). f is bijective

(ii). f is continuous and

(iii). f^{-1} is continuous.

A property $\wp$ of a metric space (X, d) is said to be ***topological property*** if it is preserved by a homeomorphism.

Theorem 4.2: Connectedness of a metric space is a topological property.

Proof: Since connectedness is preserved under a continuous function and a homeomorphism is a continuous function, therefore, connectedness is preserved under homeomorphism and hence is a topological property. ■

Theorem 4.3: Let (X, d) and (Y, ρ) be homeomorphic metric spaces. Then (X, d) is connected if and only if (Y, ρ) is connected.

Proof: Let $f : (X, d) \to (Y, \rho)$ be a homeomorphism from (X, d) onto (Y, ρ). Let (X, d) be connected. Since f is continuous and onto and since continuous image of a connected metric space is connected, therefore $f(X) = Y$ is connected. Again since f is homeomorphism, f is one to one and f^{-1} is continuous, therefore by the same argument, $X = f^{-1}(Y) = f^{-1}\big(f(X)\big)$ is connected. ■

5. Unions, Intersections of Connected Subsets

In this Section we study the behavior of connectedness with respect to the union and intersection of a family of connected sets. Later, we shall study their behavior with respect to product also. We observe that union of two connected subspaces in general may not be connected. However, if they have non – empty intersection, then their union is connected. But intersection of two connected subspaces may not be connected even if they have non – empty intersection.

Definition 5.1: Let X be any non-empty set. A family $\mathcal{B}$ of non - empty subsets of X is called a **chained collection** if, and only if for each $A, B \in \mathcal{B}$ there exists an ordered n-tuple $(U_1, U_2, \dots, U_n)$ of members of $\mathcal{B}$ with $A = U_1$, $B = U_n$ and $U_{i-1} \cap U_i \neq \phi \ \forall \, i,\ 1 < i \leq n$ such an n-tuple $(U_1, U_2, \dots, U_n)$ is called a chain from A to B in $\mathcal{B}$.

Theorem 5.1: Suppose (X, d) is a metric space and suppose $\mathcal{H}$ is a chained collection of connected subsets of X. Then $\cup\mathcal{H}$ is also connected.

Proof: Let $S = \cup\mathcal{H}$. We show that S is connected. Let $f: S \to (Y, d^*)$ be any continuous function where $Y = \{0, 1\}$ and d^* is a discrete metric on Y. Since every member of $\mathcal{H}$ is connected, we have that $f|_C$ is constant for each $C \in \mathcal{H}$. Now let $x, y \in S$ and let $A, B \in \mathcal{H}$ be such that $x \in A$ and $y \in B$. If A $= B$, then, since $f|_A$ is constant, we have $f|_A(x) = f|_A(y) = k$, where k is a constant. But this gives $f(x) = f(y) = k$ i.e. f is constant since x and y are arbitrary elements of S.

Next, if $\mathrm{A} \neq \mathrm{B}$, $\exists\, n \in \mathbb{N} - \{1\}$ and a chain $(\mathrm{U_1}, \mathrm{U_2}, \dots, \mathrm{U_n})$ of elements of $\mathcal{H}$ satisfying $\mathrm{A} = \mathrm{U_1}$, $\mathrm{B} = \mathrm{U_n}$ and $\mathrm{U_{i-1}} \cap \mathrm{U_i} \neq \emptyset$ for all $\mathrm{i} = 2,3, \dots, \mathrm{n}$.

Choose $\mathrm{u_i} \in \mathrm{U_{i-1}} \cap \mathrm{U_i}$ for all $\mathrm{i} = 2,,3, \dots, \mathrm{n}$ ... ... (1)

Since each $\mathrm{U_i}$ is connected and $\mathrm{f}|_{\mathrm{U_i}}$ is continuous and therefore constant for all $\mathrm{i} = 2,,3, \dots, \mathrm{n}$.

From (1), $u_2 \in \mathrm{U_1} \cap \mathrm{U_2} \Rightarrow \mathrm{u_2} \in \mathrm{U_1} = \mathrm{A}$.

Therefore $x, u_2 \in A \Rightarrow f(x) = f(u_2)$. Now $u_i \in U_{i-1} \cap U_i$ and $u_{i-1} \in U_{i-2} \cap U_{i-1}$ and so both u_{i-1} and $u_i \in U_{i-1}$ which implies $f(u_{i-1}) = f(u_i)\ \ \forall\ i = 3, \dots, n$.

Again from (1), we have $\mathrm{u_n} \in \mathrm{U_{n-1}} \cap \mathrm{U_n} \Rightarrow \mathrm{u_n} \in \mathrm{U_n} = \mathrm{B}$ and both $u_n, y \in B$ Hence, $f(u_n) = f(y)$. Whence, we get

$$f(x) = f(u_2) = f(u_3) = \dots = f(u_n) = f(y) \Rightarrow f(x) = f(y).$$

Since x and y are arbitrary elements of S, this shows that f is constant on S and therefore S is connected. ■

Corollary 5.1: In a metric space (X, d) if $\mathcal{H}$ is a collection of non – empty connected subsets of X for which $\cap\mathcal{H} \neq \emptyset$, then $\cup\mathcal{H}$ is connected.

Proof: Given that $\cap\mathcal{H} \neq \emptyset$. Hence for any $A, B \in \mathcal{H}$, we have $A \cap B \neq \phi$ and so $\mathcal{H}$ is a chained collection consisting of connected subsets of X and so by theorem 5.1, $\cup\mathcal{H}$ is connected. ■

Remark 5.1: The condition of non-empty intersection in the Corollary 5.1 cannot be dropped as can be seen from the following example :

Example 5.1: Let (R, d) be a metric space equipped with the standard product metric d. Let $A = \{x \in R : x < 0\}$ and let $B = \{x \in R : x > 0\}$ be subsets of R. Then A and B being respectively left ray $(-\infty, 0)$ and right ray $(0, \infty)$ are connected subsets of R. Now the set $X = A \cup B$ is expressible as a union of two disjoint open subsets namely, A and B (since $A \cap B = \phi$). Hence X is disconnected. Thus union of two connected sets need not be connected if the two connected sets do not intersect.

Remark 5.2: As we have seen in corollary 5.1, union of two or more connected spaces having non empty intersection is connected. However, the same is not true for intersection of connected spaces even if they intersect. This is illustrated in the following examples:

Example 5.2: Let $A = \{(x, 0) : -1 \leq x \leq 1\}$ be the segment on the $x-$ axis and $B = \{(x, y) : x^2 + y^2 = 1\}$ be the circle in the $xy-$ plane, then each of A and B is connected. Now, $A \cap B = \{(-1, 0), (1, 0)\}$, which is obviously disconnected.

Example 5.3: Let $A = \{(x, y) : x^2 + y^2 = 4\}$ be a circle in $\mathbb{R}^2$ and $B = \{(x, y) : y = 1\}$ be a straight line in $\mathbb{R}^2$. Then both A and B are connected. The intersection $A \cap B = \{(-\sqrt{3}, 1), (\sqrt{3}, 1)\}$ is disconnected.

Example 5.4: Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ denotes the circle of unit radius in $\mathbb{R}^2$ and $\mathrm{B} = \{(\mathrm{x}, \mathrm{y}) \in \mathbb{R}^2 : 4\mathrm{x}^2 + \mathrm{y}^2 = 1\}$ be the ellipse in $\mathbb{R}^2$. Both A and B are connected, however $\mathrm{A} \cap \mathrm{B} = \{(0, -1), (0, 1)\}$ is disconnected.

Let us now, give an alternative proof of Theorem 3.1 using Corollary 5.1 of Theorem 5.1

Alternative Proof of Theorem 3.1
Let us first assume that X is connected. Now let $x, y \in X$ be any pair of points in X. Then obviously there exist a connected subset namely X itself which contains both x and y. **Conversely**, suppose for each pair of points $x, y \in X$, there exists a connected set C containing both x and y and is such that $C \subseteq X$. Now let us fix a point $p \in X$. Then by hypothesis, for each $x \in X$, there exists a connected set C_x of X such that $\{p, x\} \subseteq C_x$. Let $\mathcal{F} = \{C_x : x \in X\}$ is a collection of connected subsets of X such that $p \in \prod_{x \in X} C_x$ *i.e.*, $\prod_{x \in X} C_x \neq \phi$. Hence, by Corollary 5.3, $\coprod_{x \in X} C_x$ is connected subset of X. Further, since $x \in C_x \ \forall x \in X$, we have $X \subseteq \coprod_{x \in X} C_x$. But this gives $X = \coprod_{x \in X} C_x$ which shows that X is connected.

6. Connected Components in Metric Spaces

Definition 6.1: Suppose (X, ρ) is a metric space. A subset $K \subseteq X$ is called a ***connected component of X*** if K is connected and there is no connected subset L in X such that K is properly contained in L. In other words, if V is any connected subset of X such that $\subseteq V \subseteq X$, then $K = V$.

If (X, ρ) is a connected metric space, then there is only one connected component namely, the set X itself. However, if (X, ρ) is disconnected, it may have several connected components. For example, if (X, ρ) is a discrete metric space then its only connected components are the singleton subsets of X.

Example 6.1: Let $X = [0, 1] \cup [2, 3] \cup [4, 5]$ be any subset of $\mathbb{R}$. Then it has only 3 connected components namely, $[0, 1]$, $[2, 3]$, and $[4, 5]$.

Example 6.2: Consider the subset

$$X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

of $\mathbb{R}$ and let ρ be the induced metric on X from $(\mathbb{R}, d)$. Then $\{0\}$ is a connected subset of (X, ρ). We claim that $\{0\}$ is a connected component in (X, ρ). Let C_0 be any connected subset in X such that $0 \in C_0$. We claim that $C_0 = \{0\}$. Suppose on the contrary, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} \in C_0$. Then for any irrational number α such that $0 < \alpha < \frac{1}{m}$, the sets $(-\infty, \alpha) \cap C_0$, $(\alpha, \infty) \cap C_0$ are nonempty, disjoint and open subsets of X whose union is C_0 This implies that C_0 is disconnected contradicting the fact that C_0 is connected. Hence $C_0 = \{0\}$.

Example 6.3: Let $\mathbb{Q}$ be the subset of rational numbers in $(\mathbb{R}, d)$. For each $x \in \mathbb{Q}$, the connected component containing x is the set $\{x\}$ itself. Equivalently, any subset S of $\mathbb{Q}$ containing more than one element is disconnected. This can be seen as follows:

Let $x, y \in S$ and let $x < y$. Let α be an irrational number such that $x < \alpha < y$ then $(-\infty, \alpha) \cap S$ and $(\alpha, \infty) \cap S$ provide a disconnection of S. This shows that S is disconnected.

Theorem 6.1: Suppose (X, ρ) is a metric space. And let $\mathcal{C}$ denote the collection of all connected components of X. Then

(i) $\forall A, B \in \mathcal{C}$ *with* $A \neq B$, $A \cap B = \phi$

(ii) $\forall A \in \mathcal{C}$,A is closed

(iii) X is the union of elements of $\mathcal{C}$.

Proof: (i). Suppose there exist $A, B \in \mathcal{C}$ such that $A \neq B$ and $A \cap B \neq \phi$. Then $A \cup B$ is connected and is such that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. But since A and B are connected components, we have $A = A \cup B = B$, a contradiction to $A \neq B$.

(ii). Let $A \in \mathcal{C}$. Then A is a connected component of X. Since the closure of a connected set is connected, therefore $\bar{A}$ is connected. Therefore have $A = \bar{A}$, i.e. A is closed.

(iii). We show that $X = \cup\{E : E \in \mathcal{C}\}$. Let $x \in X$. Let

$$C_x = \{A \subseteq X : A \textit{ is a connected subset of } X \textit{ and } x \in A\}.$$

Then $C_x \neq \phi$ as $\{x\} \in C_x$. Now $\forall A \in C_x$, $x \in A$ and so we have $x \in \prod_{A \in C_x} A$ *i.e.,* $\prod_{A \in C_x} A \neq \phi$.
Therefore $\coprod_{A \in C_x} A$ is connected.
Let $Z = \coprod_{A \in C_x} A$. Then Z is connected and $x \in Z$. We claim that Z is a connected component in X. Let S be any connected subset of X such that $x \in S$ and $Z \subseteq S$. But since $x \in S$ and is a connected subset of X, therefore $S \in C_x$ and so $S \subseteq Z$. Hence $Z = S$. Therefore, Z is a connected component in X containing x. Whence, $Z \in \mathcal{C}$. Thus we see that for each $x \in X$, there exists $E \in \mathcal{C}$ such that $x \in E$. Thus $X \subseteq \cup\{E : E \in \mathcal{C}\} \subseteq X$. Thus $X = \cup\{E : E \in \mathcal{C}\}$. ■

Consider the metric space (X, ρ), where $X = (0,1) \cup (2,3) \cup (4,5)$ and ρ is the metric induced on X by the usual metric of $\mathbb{R}$. Then clearly, X has 3 connected components namely, $(0,1)$, $(2,3)$ and $(4,5)$ which are open as well closed.

Now to see that connected components need not be open, consider subset

$$X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

of $\mathbb{R}$ and the induced metric ρ on X from $(\mathbb{R}, d)$. Then we have already seen that $\{0\}$ is a connected component in (X, ρ). However, it is easily seen that $\{0\}$ is not open in (X, d).

7. CONNECTED SUBSETS OF $(\mathbb{R}, d)$

Let $(\mathbb{R}, d)$ be a metric space where $\mathbb{R}$ denotes the set of real numbers and d denotes the usual metric defined on $\mathbb{R}$. In this section we shall study what subsets of $\mathbb{R}$ are connected under d. It is obvious that all singleton subsets of $\mathbb{R}$ are connected. However, a finite subset containing more than one point may not be connected, since if A is a finite subset of $\mathbb{R}$, then (A, d_A) would be a discrete metric which is disconnected. Similarly, the subspace (I, d_I) of integers being a discrete subspace is also disconnected . Even, we have the following:

Example 7.1: The subspace $(\mathbb{Q}, d_{\mathbb{Q}})$ of $(\mathbb{R}, d)$ is disconnected.

Let $z \in \mathbb{R}$ be such that z is an irrational number. Then $(-\infty, z)$ and (z, ∞) are open subsets in $\mathbb{R}$. Therefore, $(-\infty, z) \cap \mathbb{Q}$ and $(z, \infty) \cap \mathbb{Q}$ are open subsets in $\mathbb{Q}$ satisfying

$\mathbb{Q} = ((-\infty, z) \cap \mathbb{Q}) \cup ((z, \infty) \cap \mathbb{Q})$ and $((-\infty, z) \cap \mathbb{Q}) \cap ((z, \infty) \cap \mathbb{Q}) = \emptyset$ implying that $(\mathbb{Q}, d_{\mathbb{Q}})$ is disconnected. ■

Example 7.2: The subspace $(\mathbb{R}\sim\mathbb{Q}, d_{\mathbb{R}\sim\mathbb{Q}})$ of $(\mathbb{R}, d)$ is disconnected. (*Prove yourself*)

Theorem 7.1: A subset S of $(\mathbb{R}, d)$ is connected if and only S is an interval.

Proof: Let S be a connected subset of $(\mathbb{R}, d)$. We show that S is an interval. If $S = \emptyset$, then the result is obvious. So let us assume that $S \neq \emptyset$. Suppose that S is not an interval. Then there exist real numbers x, y, and z such that $x < z < y$ where $x, y \in S$ and $z \notin S$. Define $A_z = (-\infty, z) \cap S$ and $B_z = (z, \infty) \cap S$. Then A_z and B_z are open subsets of S and are such that $S = A_z \cup B_z$ and $A_z \cap B_z = \emptyset$ implying that S is disconnected, a contradiction.

Conversely, let S be a subset of $\mathbb{R}$ such that S is an interval. We show that S is connected. We show that each continuous function $f : S \to \{0, 1\}$ is constant. Let $f : S \to \{0, 1\}$ be continuous and let f be not constant. Then there exist at least one pair of points x_1 and y_1 in S such that $f(x_1) = 0$ and $f(y_1) = 1$. Since x_1 and y_1 are real numbers, without any loss of generality, we may assume that $x_1 < y_1$. Let $a = \frac{x_1+y_1}{2}$ be the midpoint of x_1 and y_1. Then either $f(a) = 0$ or $f(a) = 1$. If $f(a) = 0$, let us put $a = x_2$ and $y_2 = y_1$. Now,

$$y_2 - x_2 = y_1 - \frac{x_1 + y_1}{2} = \frac{y_1 - x_1}{2}.$$

On the other hand If $f(a) = 1$, we put $x_2 = x_1$ and $a = y_2$. Also,

$$y_2 - x_1 = a - x_1 = \frac{x_1 + y_1}{2} - x_1 = \frac{y_1 - x_1}{2}.$$

By the above construction, we have $x_1 \leq x_2 \leq y_2 \leq y_1$.

Continuing this process indefinitely, we get monotonic increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ and monotonic decreasing sequence $\{y_n\}_{n\in\mathbb{N}}$ with such that:

(i). $x_i \leq y_j \quad \forall\, i, j \in \mathbb{N}$

(ii). $y_n - x_n = \frac{1}{2^{n-1}}(y_1 - x_1)$

(iii). $f(x_n) = 0$ and $f(y_n) = 1 \quad \forall n \in \mathbb{N}$.

The sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are bounded above and bounded below, respectively and hence are convergent. Let $\{x_n\}_{n\in\mathbb{N}} \to x$ and let $\{y_n\}_{n\in\mathbb{N}}$ \to y. Since f is continuous, we have $f(x) = 0$ and $f(y) = 1$. Further, we have that

$$\begin{aligned} &\lim_{n\to\infty}(y_n - x_n) = \lim_{n\to\infty}\frac{1}{2^{n-1}}(y_1 - x_1) = 0 \\ \Rightarrow\ &\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n \\ \Rightarrow\ & x = y\,. \end{aligned}$$

If we denote by z this common value, i.e. if $z = x = y$, we see that $f(z) = f(x) = 0$ and $f(z) = f(y) = 1$ contradicting that f is a function. This contradiction proves that f is constant and so S is connected. ■

Corollary 7.1: The range of a continuous real valued function defined on a connected metric space is an interval.

Proof: Let (X, d) be a connected metric space and let f be a continuous real valued function on X. Then $f(X)$ is a connected subset of R and so by Theorem 3.1 is an interval. ■

Example 7.3: All intervals of the form (a,∞), $(-\infty,a)$, $[a,\infty)$, $(-\infty,a]$, $[a,b]$, $(a,b]$, $[a,b)$, (a,b) are connected.

The following theorem is also a consequence of theorem 3.1 called the Intermediate Value Theorem:

Theorem 7.2: Let $f : (X,\rho) \to (\mathbb{R}, d)$ be a continuous function from a connected metric space (X,ρ) into the metric space $(\mathbb{R}, d)$. Then for any $x, y \in X$ and $s \in R$ such that $f(x) < s < f(y)$ there exists $z \in X$ such that $f(z) = s$.

Proof: Since (X,ρ) is a connected metric space and since f is a continuous function, therefore $f(X)$ is a connected subset of $\mathbb{R}$. Hence $f(X)$ is an interval. Now since $f(x), f(y) \in f(X)$ and $s \in R$ such that $f(x) < s < f(y)$, therefore, there exists an element $z \in X$ such that $f(z) = s$ which proves the theorem. ■

Before we proceed further, let us discuss two important results concerning connected metric spaces.

Theorem 7.3: Every line segment in $\mathbb{R}^2$ is connected

Proof: Let AB be a line segment in $\mathbb{R}^2$. Let us take point A to origin and the line segment AB produced as the x –axis. Let it be denoted by AX. Now draw a perpendicular AY from A on AX. Let the coordinates of the point B be $(b, 0)$. Now define a function

$$f : [0,b] \to \mathbb{R}^2 \quad by \quad f(x) = (x, 0).$$

We claim that f is continuous.

Let x_0 be any arbitrary element of $[0,b]$ and let $\{x_n\}$ be a sequence in $[0,b]$ such that $\{x_n\} \to x_0$. We show that $\{f(x_n)\} \to f(x_0)$. Now $f(x_n) = (x_n, 0) \to (x_0, 0) = f(x_0)$. Hence $f(\{x_n\}) \to f(x_0)$ implying that f is continuous. Now since $[0,b]$ is connected and a continuous image of a connected space is connected. Therefore, $f([0,b])$ is a connected subset of $\mathbb{R}^2$. But since $f([0,b]) = AB$, therefore AB is connected. ■

Theorem 7.4: Every circle is a connected subset of $\mathbb{R}^2$.

Proof: Let C be a circle of radius r having centre at the origin. Then the parametric equation of C is given by $= r\cos\theta$, $y = r\sin\theta$ where $0 \leq \theta \leq 2\pi$. Now define a function $f : \mathbb{R} \to \mathbb{R}^2$ by setting $f(\theta) = (r\cos\theta, r\sin\theta)$. We show that f is continuous. Let θ_0 be any element of $\mathbb{R}$. Let d denote the product metric on $\mathbb{R}^2$ which is nothing but the Euclidean metric. Hence for any $\theta \in \mathbb{R}$, we have

$$d\big(f(\theta), f(\theta_0)\big) = d\big((r\cos\theta, r\sin\theta), (r\cos\theta_0, r\sin\theta_0)\big)$$
$$= \sqrt{(r\cos\theta - r\cos\theta_0)^2 + (r\sin\theta - r\sin\theta_0)^2}$$
$$= 2r\left|\sin\frac{1}{2}(\theta - \theta_0)\right|$$

Since the sine function is a continuous function, therefore given ε> 0 we can find a δ> 0 such that whenever, $\left|\frac{1}{2}(\theta-\theta_0)\right| < \frac{1}{2}\delta$ we have $\left|\sin\frac{1}{2}(\theta-\theta_0)\right| < \frac{\epsilon}{2r}$

i.e., $|\theta - \theta_0| < \delta$ we have $d\big(f(\theta), f(\theta_0)\big) < \epsilon$.

This shows that f is continuous at θ_0. But θ_0 is an arbitrary point of $\mathbb{R}$, therefore f is continuous on $\mathbb{R}$. Since continuous image of a connected space is connected and since $[0, 2\pi[$ is a connected subset of $\mathbb{R}$, therefore its image the circle C is connected, proving the theorem. ■

8. Product of Connected Metric Spaces

In this section we study the behavior of connectedness with respect to the product of metric spaces. We have the following:

Theorem 8.1: Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Let (X, d) denote the product metric space where $X = X_1 \times X_2$ and $= d_1 \times d_2$. Then (X, d) is connected if and only if each coordinate space is connected.

Proof: Let first assume that (X, d) is connected. Since the projection functions p_1 and p_2 from (X, d) onto coordinate spaces, i.e.,

$P_1 : (X, d) \to (X_1, d_1)$ defined by $P_1(x, y) = x \;\; \forall\, x \in X_1, y \in X_2$

And $P_2 : (X, d) \to (X_2, d_2)$ defined by $P_2(x, y) = y \;\; \forall\, x \in X_1, y \in X_2$

are continuous and since a continuous image of a connected space is connected and therefore both (X_1, d_1) and (X_2, d_2) are connected spaces.

Conversely, let us assume that both (X_1, d_1) and (X_2, d_2) are connected metric spaces. We show that (X, d) is a connected metric space. Let (x, y) and (x_1, y_1) be any pair of points in $X_1 \times X_2$. We show that there is a connected subset of $X_1 \times X_2$ that contains both the points

Since $\{x\} \times X_2$ is a copy of the connected metric space (X_2, d_2) and therefore is a connected subset of $X_1 \times X_2$ containing the points (x, y) and (x, y_1). Similarly, $X_1 \times \{y_1\}$ is a copy of the connected metric space (X_1, d_1) and hence is a connected subset of $X_1 \times X_2$containing points (x, y_1) and (x_1, y_1). We thus have two connected subsets $\{x\} \times X_2$ and $X_1 \times \{y_1\}$ such that

$(\{x\}\times X_2)\cap(X_1\times\{y_1\})\neq\phi$. Hence $(\{x\}\times X_2)\cup(X_1\times\{y_1\})$ is a connected subset of (X,d) containing both the points (x,y) and (x,y_1) proving that (X,d) is connected by Theorem 2.11. ■

9. TOTALLY DISCONNECTED METRIC SPACES

In a discrete metric space (X,ρ), we have seen that the singleton subsets are the only connected subsets of X and so are also the connected components. This observation has lead us to introduce an important class of spaces known as the totally disconnected spaces.

Definition 9.1: A metric space (X,ρ) is said to be totally disconnected if its connected components are all singleton sets .

As mentioned above, all discrete metric spaces are totally disconnected. However a totally disconnected metric space may fail to be discrete as shown in the following example:

Example 9.2: Consider $(\mathbb{Q},d_{\mathbb{Q}})$ the metric space of rational numbers equipped with the metric $d_{\mathbb{Q}}$ from the usual metric. Then the open subsets of $\mathbb{Q}$ are of the form $G\cap\mathbb{Q}$, where G is open in $\mathbb{R}$. None of these is a singleton set, and so $\mathbb{Q}$ is not discrete.

But $\mathbb{Q}$ is totally disconnected because every connected component of $\mathbb{Q}$ is a connected subset of $\mathbb{R}$ and is therefore an interval. However, the only intervals that are subsets of $\mathbb{Q}$ are of the form $[a,a]=\{a\}$ for all $a\in\mathbb{Q}$. Thus the only component of $\mathbb{Q}$ are singletons only. This shows that $(\mathbb{Q},d_{\mathbb{Q}})$ is totally disconnected.

Theorem 9.1: The Cantor set C is totally disconnected under induced metric defined on $[0,1]$.

Proof: Let us recall the construction of the Cantor's set. As a first step, We divide the interval $[0,1]$ in three equal parts and exclude the interval $\left(\frac{1}{3},\frac{2}{3}\right)$. The number of the closed subintervals left is 2 and the length of each of these intervals is 3^{-1} . Moreover, they are disjoint. Let the union of these two intervals be denoted by A_1. In the second step we divide each of the remaining two closed intervals $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$ into three equal parts and again we remove the middle third open intervals from them. Now the number of remaining closed intervals is 2^2 and the length of each of these closed intervals is 3^{-2}. Also, the remaining closed intervals are pairwise disjoint. Let the union of these four intervals be denoted by A_2. We continue this process indefinitely. After k^{th} step, we see that A_k is the union of 2^k pairwise disjoint closed subintervals each of which is of length 3^{-k}. The cantor set C is defined as the intersection of all A_k's for uncountably many k.

We claim that the only connected subsets of C are singletons. Let K be a non - empty connected subset of C. Then $K\subset A_k$ for each k. Since A_k is the union of 2^k Pairwise disjoint closed subintervals each of length 3^{-k}, therefore, K is contained in exactly one of these subintervals for otherwise K would be disconnected. Hence $diam\,K\leq 3^{-k}$ for all $k\geq 0$. Hence for sufficiently large k, $diam\,K=0$ this implies that K is singleton. ■

10. PATH WISE CONNECTED METRIC SPACES

Definition 10.1: Let (X,ρ) be a metric space. Then a continuous function $f:[0,1] \to (X,\rho)$ from the closed interval $[0,1]$ into (X,ρ) is called a path in X from the point $f(0)$ to the point $f(1)$. The points $f(0)$ and $f(1)$ of X are called the endpoints of the path.

Definition 10.2: A metric space is said to be path wise connected if for each pair of points $x, y \in X$ there exists a continuous function $f:[0,1] \to (X,\rho)$ such that $f(0) = x$ and $f(1) = y$. In other words, there exists a path in X with end points x and y.

THEOREM 10.3: Every path is uniformly continuous and its image is connected and compact.

Proof: Let (X,ρ) be a metric space and let f be a continuous function from $[0,1]$ onto (X,ρ). Since every continuous function defined on a compact space is uniformly continuous therefore, $[0,1]$ being compact, f is uniformly continuous. Further, since continuous image of a compact and connected space respectively is compact and connected, therefore, $[0,1]$ being both compact and connected, image of the path is both compact And connected. ■

A useful reformulation of path wise connectedness is given in the following proposition.

Theorem 10.4: Let (X,ρ) be a metric space and $x_0 \in X$ be any element. (X,ρ) is Path wise connected if and only if each $x \in X$ can be joined to x_0 by a path.
Proof: If (X,ρ) is path wise connected, then the condition of the theorem trivially holds. Conversely, let assume that the condition of the theorem is satisfied and let a, b be any elements of X. By hypothesis, let $f:[0,1] \to (X,\rho)$ be a path from a to x_0 and let $g:[0,1] \to (X,\rho)$ be a path from x_0 to b. Now define a function $h:[0,1] \to (X,\rho)$ as follows :

$$h(x) = \begin{cases} \mathrm{f}(2\mathrm{x}) & 0 \le \mathrm{x} \le \frac{1}{2} \\ \mathrm{g}(2\mathrm{x}-1) & \frac{1}{2} < \mathrm{x} \le 1 \end{cases}.$$

Now since $\lim_{x \to \frac{1}{2}-} h(x) = f(1) = x_0$, $\lim_{x \to \frac{1}{2}+} h(x) = g(0) = x_0$ thus we see that h is a continuous function satisfying $h(0) = a$ and $h(1) = b$. Since a and b are arbitrary elements of X, we conclude that (X,ρ) is path wise connected. ■

The general relation between connectedness and path wise connectedness is given by the following theorem.

Theorem 10.5: Let (X,ρ) be a metric space. If (X,ρ) is path wise connected, then it is connected.

Proof: Suppose (X,ρ) is path wise connected . If (X,ρ) is empty, then the result is obvious so let us assume that $\neq \phi$. Let $a \in X$; then for each $b \in X$, there exists a continuous function $f_b:[0,1] \to (X,\rho)$ such that $f_b(0) = a$ and $f_b(1) = b$. Since $[0,1]$ is connected and f_b is continuous, $f_b([0,1])$ is connected. Moreover, $a \in \prod_{b\in X} f_b([0,1])$ and, hence, by Corollary 5.1 of Lesson I, $X = \coprod_{b\in X} f_b([0,1])$ is connected in X. ■

Theorem 10.6: Every continuous image of a path wise connected metric space (X,ρ) is path wise connected.

Proof: Let $f : (X,\rho) \to (Y,d)$ be a continuous function defined on a path wise metric space (X,ρ) onto a metric space (Y,d). We show that $f(X) = Y$ is path wise connected. Let $a, b \in f(X)$ then there exist $x\,z \in X$ such that $f(x) = a$ and $f(z) = b$. Now since (X,ρ) is path wise connected, there exists a continuous function $g : [0,1] \to (X,\rho)$ such that $g(0) = x$ and $g(1) = z$. Consider the composition function $f \circ g : [0,1] \to (Y,d)$. $f \circ g$ being a composition of two continuous functions is a continuous function and satisfies $f \circ g\,(0) = a$ and $f \circ g\,(1) = b$. Since a and b are arbitrary elements of $f(X) = y$, therefore (Y,d) is path wise connected. ■

Next, we show that path wise connectedness of a metric space is stronger condition than its being connected. In support of this statement it is sufficient to give an example of a connected metric space that is not path wise connected.

Example 10.7: Let

$$A = \left\{ \left(x, sin\left(\frac{1}{x}\right)\right) \in \mathbb{R}^2 : 0 < x \le \left(\frac{1}{\pi}\right)\right\}$$

and let $B = A \cup \{(0,0)\}$.
Since the set $(0,1/\pi)$ being an interval is a connected set, and since the function $f : (0,1) \to A$ defined by $f(x) = (x, sin(1/x))$ is continuous that maps this interval onto the set A, it follows that A is connected . Now if we can show that $(0,0)$ is a closure point of , then from A $\subset$ B $\subseteq \overline{\text{A}}$, it will immediately follow that B is connected.
Now to show that $(0,0)$ is a closure point of A, it suffices to find a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in A such that $\lim_{n\to\infty} x_n = (0,0)$. But if $x = 1/(n\pi)$, where n $\in$ N, then $sin(1/x) = sin(n\pi) = 0$, and so the point $x_n = (1/(n\pi), 0)$ is in the set A. Moreover, it is clear that $x_n \to (0,0)$ as $n \to \infty$, as required. B is thus connected. However, it is not path wise connected.

11. SOME SOLVED PROBLEMS

Problem 11.1: State with justification as to which of the following sets in R^2 equipped with the standard metric are connected:

(i) $A = \{(x,y) : x \le 1\}$

(ii) $S = \{(x,y) : y = 0 \; when \; x \le 2;\; y = 1 \; otherwise\}$;

(iii) $B = \{(x,y) : y = sin\,x\}$;

(iv) $T = \{f(x,y) : 1 < x^2 + y^2 \le 2\}$;

(v) $D = \{(x,y) : 0 < x < 1 \; and \; y \in \mathbb{Q}\}$.

Solution: (i) The set A is connected since any two points in A can be joined by a straight line segment.

(ii) The set S is not connected because of the following argument:
Let $G = \{(x,y) : y < 1/2\}$ and $H = \{(x,y) : y > 1/2\}$. Then G and H are open subsets in $\mathbb{R}^2$. Therefore $S \cap G$ and $S \cap H$ are non empty subsets of S satisfying $S = (S \cap G) \cup (S \cap H)$ with $(S \cap G) \cup (S \cap H) = \phi$. Thus $(S \cap G)$ and $(S \cap H)$ form a separation of S.
(iii) The set B is connected. Looking at the graph of the function $sin\,x$ it is clear that any two points in the set can be joined by the continuous curve $(z) = \sin z$.

(iv) The set T is connected, since any two points on the curve can be joined by a continuous curve.

(v) The set D is not connected for the following reason:
Let $G = \{(x,y) : y < \sqrt{2}\}$ and $H = \{(x,y) : y > \sqrt{2}\}$. Then G and H are disjoint open subsets of $\mathbb{R}^2$. Therefore $D \cap G$ and $D \cap H$ are disjoint open subsets of D such that $D = (D \cap G) \cup (D \cap H)$. This shows that $D \cap G$ and $D \cap H$ form a separation of D. ■

Problem 11.2: Let (X,d) be a metric space and let G be an open subset of X and be such that G is expressible as a union of two separated sets A and B. Then each of A and B is open.
Solution: Let G be any open subset of X. Further, let us assume that $G \neq \phi$. It is given that $G = A \cup B$ with $A \cap \bar{B} = \bar{A} \cap B = \phi$. Now if $x \in A$, then we have that $x \in G$ and $x \notin \overline{\mathrm{B}}$. Since G is an open set and $x \in G$, there exists $r_1 > 0$ such that $x \in B(x, r_1) \subseteq G$. Also, since $x \notin \overline{\mathrm{B}}$, there exists $r_2 > 0$ such that $x \in B(x, r_2)$ and $B(x, r_2) \cap B = \phi$.
Now, let $= min\{r_1, r_2\}$. Then $x \in B(x,r) \subseteq A$ implying that A is a neighborhood of x. Since x is any arbitrary element of A, we conclude that A is a neighbourhood of each of its points. Therefore A is open. Proceeding exactly on the same line as above, we can show that B is also open. ■

Problem 11.3: Let (X,d) be a metric space and let X have a finite number of components $C_1, C_2, \dots, C_k$. Then each component in X is open.
Solution: We know that components of a metric space (X,d) are closed and connected subset of X whose union is X. Hence $C_1, C_2, \dots, C_k$ are closed connected subsets of X and are such that $= \coprod_{i=1}^{k} C_i \quad with \quad C_i \cap C_j = \phi \ \forall\, i \neq j$. Hence for each $i, 1 \leq i \leq k$, we have

$$C_i = X - \coprod_{\substack{j=1 \\ j \neq i}}^{k} C_j \ .$$

Now since a finite union of closed sets is closed and complement of a closed set is open, therefore C_i is open. ■

Problem 11.4: Let $f : \mathbb{R} \to \mathbb{R}$ be any function defined as

$$f(x) = \begin{cases} f(x) \in \mathbb{R} - \mathbb{Q} & if \ \ x \in \ \mathbb{Q} \\ f(x) \in \mathbb{Q} & if \ \ x \in \ \mathbb{R} - \mathbb{Q} \end{cases},$$

then f cannot be continuous.
Solution: Let if possible f be continuous. Then since $\mathbb{R}$ is connected and the continuous image of a connected space is connected, therefore, $f(\mathbb{R})$ is connected. Since $f(\mathbb{R})$ is connected subset of $\mathbb{R}$, therefore $f(\mathbb{R})$ is either a singleton set or is an interval. However, the way the function f is defined, $f(\mathbb{R})$ is not a singleton, i.e., f is not constant and so $f(\mathbb{R})$ is an interval. Now $f(\mathbb{R})$ being an interval is uncountable. But we are given that $f(\mathbb{Q}) \subseteq \mathbb{R} - \mathbb{Q}$ and $f(\mathbb{R} - \mathbb{Q}) \subseteq \mathbb{Q}$. This implies that $f(\mathbb{R}) = f\big(\mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})\big) = f(\mathbb{Q}) \cup f(\mathbb{R} - \mathbb{Q})$ is countable which contradicts the fact that $f(\mathbb{R})$ is uncountable. This contradiction proves that f cannot be continuous. ■

Problem 11.5: Let A and B be connected subsets of a metric space (X,d) and let A contain a point of $\bar{A}$, then $A \cup B$ is connected.

Solution: Suppose that $x \in \overline{B} \cap A$. Since $B \subset \{x\} \cup B \subseteq \overline{B}$, and since B is connected, it follows that $\{x\} \cup B$ is connected. Furthermore, since $A = A \cup \{x\}$, we have that

$$A \cup B = (A \cup \{x\}) \cup B = A \cup (\{x\} \cup B).$$

This shows that $A \cup B$ is expressed as the union of connected sets A and $\{x\} \cup B$, which have nonempty intersection (since both sets contain x). Hence, $A \cup B$ is connected.

12. Summary

In this lesson, we have defined the notion of connectedness and proved several criteria for connectedness of metric spaces. We showed that it is a topological property. Later, we discussed subspaces of connected metric spaces. We showed that the union of a collection of connected spaces having non empty intersection is connected. However, the intersection of two connected subsets may fail to be connected even if they intersect.

We have seen in the previous lesson, that in any metric space (X, d), singletons are always connected irrespective of whether the metric space is connected or disconnected. However, the problem is whether in a disconnected metric space there exist connected subsets containing more than one element. In this lesson, we considered this problem and introduced the notion of connected components, that is the maximal connected subsets of a metric space. We proved that they are always closed. However, if the number of components in a metric space is finite, then they are open as well.

We further showed that the collection of components form a partition of any metric space. Later, we discussed the metric space (R, d) and showed that it is always connected. Also, we proved that a subspace of R is connected if and only if it is an interval.

We close this lesson by discussing two variants of connectedness, namely, Totally disconnectedness and path wise connectedness. We showed that path wise connectedness is stronger than connectedness and it is also preserved under homeomorphism. As far as total disconnectedness is concerned, we have proved that its components are only singletons, however its converse is not true.

13. Problems for Practice

1. Which of the following subsets of $(\mathbb{R}, d)$ are Connected?
   (i). The set of integers
   (ii). The set of rational numbers
   (iii). $[a, b]$
   (iv). $\mathbb{R} - [a, b]$
2. Show that a metric space (X, ρ) is totally disconnected if for each pair of points $x, y \in X$, there exist A and B, $A \neq \phi, B \neq \phi, X = A \cup B, A \cap B \neq \phi, A \cap B \neq \phi$ such that $x \in A$ and $y \in B$.
3. Show that the set of all irrational numbers under the induced metric from the usual metric on $\mathbb{R}$ is totally disconnected.
4. Suppose $(X, d)$is a metric space and let S be a connected subset of X such that $S \subset A \cup B$ where A and B are separated subsets of X. Show that $S \subset A$ or $S \subset B$.
5. Which of the following sets are path wise connected

(a). $[0,1]$
(b). $[4,5] \cup [6,7]$
(c). $\{x \in [0,1] : x \in \mathbb{Q}\}$
(d). $\{(x,y) \in \mathbb{R}^2 : x.y \geq 1, x > 1\} \cup \{(x,y) \in \mathbb{R}^2 : x.y \leq 1, x \leq -1\}$.

6. Find two subsets A and B of $\mathbb{R}^2$ and a point $p \in \mathbb{R}^2$ such that $A \cup B$ is disconnected but $A \cup B \cup \{p\}$ is connected.
7. Give an example of a disconnected subspace T of a topological space R for which there are no nonempty sets $A, B \subset T$ such that $T = A \cup B$ and $\bar{A} \cap \bar{B} = \phi$.

8. Show that there is no continuous injective map $f : \mathbb{R}^2 \to \mathbb{R}$.

9. Consider the following Problem. The first space is considered as a subspace of R, the three latter spaces as subspaces of $\mathbb{R}^2$; in all cases $\mathbb{R}$ and $\mathbb{R}^2$ have the Euclidean metric. Decide whether these spaces are connected:
(a). $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
(b). $(\mathbb{Q} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Q})$
(c). $\{(x,y) : (x+1)^2 + y^2 < 1\} \cup \{(x,y) : (x-1)^2 + y^2 \leq 1\}$
(d). $\{(x,y) : xy \geq 1\}$

10. Give an example of a connected space Y but whose interior is not connected.

11. Suppose Y is a connected subset of a metric space (X,d). Must the boundary of Y be connected?

12. Let $A = (0,1) \cup (1,2]$ be a subset of $\mathbb{R}$. Show that A is disconnected but $\bar{A}$ is connected.
13. Give example of two disconnected spaces A and B of a metric space (X,d) such that $A \cup B$ is connected.
14. Give an example of two connected spaces A and B such that $\bar{A} \cap \bar{B} \neq \phi$ but $A \cup B$ is disconnected.
15. If A and B are connected subsets of a metric space and A contains a point of $\bar{B}$ then $A \cup B$ is connected.

16. Suppose (X,d) is a metric space and let S be a connected subset of X such that $S \subset A \cup B$ where A and B are separated subsets of X. Show that $S \subset A$ or $S \subset B$.
17. Suppose (X,d) is a metric space and let A and B subsets of X both A and B are closed in $A \cup B$. If both $A \cup B$ and $A \cap B$ are connected then both A and B are connected.
18. Let A and B be two connected subsets such that $A \cap \bar{B} \neq \phi$ and $\bar{A} \cap B \neq \phi$. Show that $A \cup B$ is connected.
19. Prove or disprove that the interior and the boundary of a connected set S in a metric space (X,d) are connected.
20. Let $A = \left\{\left(x, \sin\frac{1}{x}\right) : 0 < x \leq 1\right\} \subseteq \mathbb{R}^2$ where $\mathbb{R}^2$ is equipped with the standard metric and if $S = \overline{A}$ then show that S is connected.

14. Multiple Choice Questions:

CONNECTEDNESS

1. Let $(\mathbb{R}, d)$ be a metric space where as usual $\mathbb{R}$ is the set of real numbers and d is the usual metric on $\mathbb{R}$. Put a tick (√) on the correct statement.
   (a). Every singleton subset of $\mathbb{R}$ is connected
   (b). Every finite subset of $\mathbb{R}$ containing more than one element is connected
   (c). Every countable subset of $\mathbb{R}$ is connected
   (d). Every uncountable subset of $\mathbb{R}$ is Connected.

2. Let $(\mathbb{R}, d)$ be a metric space where as usual $\mathbb{R}$ is the set of real numbers and d is the usual metric on $\mathbb{R}$. Put a tick (√) on the correct statement.
   (a). The set $(-2, 3) - \{0\}$ is connected
   (b). The set $(2, 5) \cup \{0\}$ is connected
   (c). The set $(2, 5) \cup (5, 8)$ is connected
   (d). The set $(2, 5) \cup \{5\}$ is connected.

3. Let $(\mathbb{R}, d)$ be a metric space where as usual $\mathbb{R}$ is the set of real numbers aand d is the discrete metric on $\mathbb{R}$. Put a tick (√) on the correct statement.
   (a). The set $\mathbb{R}$ is connected
   (b). The set $\mathbb{Q}$ is connected in $\mathbb{R}$
   (c). every finite subset of $\mathbb{R}$ is connected in $\mathbb{R}$
   (d). $\mathbb{R} - \mathbb{Q}$ is connected in $\mathbb{R}$.

4. Let (X, d) be a connected metric space and let $(Y = \{0, 1\}, d^*)$ where d^* is the discrete metric on Y. Put a tick (√) on the correct statement.
   (a). Each function $f : (X, d) \to (Y, d^*)$ is continuous but non- constant.
   (b). Each function $f : (X, d) \to (Y, d^*)$ is non - constant
   (c). Each continuous function $f : (X, d) \to (Y, d^*)$ is constant.
   (d). Each function $f : (X, d) \to (Y, d^*)$ is a homeomorphism.

5. Let (X, d) be a connected metric space. Put a tick (√) on the correct statement.
   (a). X has a non - empty closed and open subset of X.
   (b). There exist a pair of non – empty disjoint open subsets G_1 and G_2 of X such that $X = G_1 \cap G_2$.
   (c). There exist a pair of non-empty disjoint closed subsets F_1 and F_2 of X such that $X = F_1 \cup F_2$.
   (d). For any subset A of , $\partial \mathrm{A} \neq \phi$.

6. Let $(\mathbb{R}, d)$ be metric space endowed with the usual metric. Let $\mathbb{Z}$ denote the set of integers. Put a tick (√) on the correct statement.
   (a). The collection of positive integers $\mathbb{Z}^+$ is connected
   (b). The set $\mathbb{Z} - \{0\}$ is connected.
   (c). The set $\mathbb{Z}$ is connected.
   (d). The set $\mathbb{Z}$ is disconnected

7. Let $(\mathbb{R}, d)$ be a metric space where as usual $\mathbb{R}$ is the set of real numbers and d is the usual metric on $\mathbb{R}$. Put a tick (√) on the correct statement.

(a). Each singleton set is a component in $\mathbb{R}$
(b). The set $(2,5) \cup \{0\}$ is a component in $\mathbb{R}$
(c). The set $\mathbb{Q}$ is a component in $\mathbb{R}$
(d). The set $\mathbb{R}$ itself is a component of $\mathbb{R}$.

8. Let $(\mathbb{R}, d)$ be a totally disconnected metric space where as usual $\mathbb{R}$ is the set of real numbers and d is a metric on $\mathbb{R}$. Put a tick (√) on the correct statement.
   (a). $(\mathbb{R}, d)$ is path wise connected
   (b). Every proper subset of $\mathbb{R}$ is connected
   (c). Every proper subset of $\mathbb{R}$ is disconnected
   (d). Each singleton subset of $\mathbb{R}$ is connected.

9. Let (X, d) be a metric space and let A be any proper non – empty subset of X such that $\partial A = \phi$. Put a tick (√) on the correct statement.
   (a). (X, d) is path wise connected
   (b). (X, d) is connected
   (c). (X, d) is disconnected
   (d). None of the above is true.

10. Let (X, d) be a metric space. Put a tick (√) on the correct statement.
   (a). Every closed subset of X is a component in X.
   (b). Every component in X is a closed subset of X
   (c). Every component in X is an open subset of X
   (d). There exists a component in X which is neither open nor closed subset of X.

11. Let (X, d) be a metric space and let $\mathcal{H}$ denote the family of all components in X. Put a tick (√) on the correct statement.
   (a). If $A, B \in \mathcal{H}$ then $A \cap B \neq \phi$.
   (b). If $A, B \in \mathcal{H}$ then $A \cup B \in \mathcal{H}$
   (c). $\cup\{A : A \in \mathcal{H}\} = X$
   (d). $\cup\{A : A \in \mathcal{H}\} \neq X$

15. References

[1]. GEORGE F. SIMMONS, INTRODUCTION TO TOPOLOGY AND MODERN ANALYSIS, McGRAW – HILL BOOK Co., NEW YORK, 1963

[2]. E.T.COPSON, METRIC SPACES, CAMBRIDGE UNIVERSITY PRESS, CAMBRIDGE, 1968.

[3]. MICHEAL O SEARCOID, METRIC SPACES, SPRINGER INTERNATIONAL EDITION, NEW DELHI, 2008.

[4]. W.A. SUTHERLAND, INTRODUCTION TO METRIC AND TOPOLOGICAL SPACES, OXFORD UNIVERSITY PRESS, OXFORD, 1975.

[5]. SATISH SHIRALI AND H L VASUDEVA, METRIC SPACES, SPRINGER VERLAGE LONDON, LIMITED, 2006.