

Metric spaces and topology

Content. Metric spaces and topology. Polish Space. Arzelá-Ascoli Theorem. Convergence of mappings. Skorohod metric and Skorohod space.

1 Metric spaces. Open, closed and compact sets

When we discuss probability theory of random processes, the underlying sample spaces and σ -field structures become quite complex. It helps to have a unifying framework for discussing both random variables and stochastic processes, as well as their convergence, and such a framework is provided by metric spaces.

Definition 1. A metric space is a pair (S, ρ) of a set S and a function $\rho : S \times S \rightarrow \mathbb{R}_+$ such that for all $x, y, z \in S$ the following holds:

1. $\rho(x, y) = 0$ if and only if $x = y$.
2. $\rho(x, y) = \rho(y, x)$ (symmetry).
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ (triangle inequality).

Examples of metric spaces include $S = \mathbb{R}^d$ with $\rho(x, y) = \sqrt{\sum_{1 \leq j \leq d} (x_j - y_j)^2}$, or $= \sum_{1 \leq j \leq d} |x_j - y_j|$ or $= \max_{1 \leq j \leq d} |x_j - y_j|$. These metrics are also called $\mathbb{L}_2, \mathbb{L}_1$ and \mathbb{L}_∞ norms and we write $\|x - y\|_1, \|x - y\|_2, \|x - y\|_\infty$, or simply $\|x - y\|$. More generally one can define $\|x - y\|_p = \left(\sum_{1 \leq j \leq d} (x_j - y_j)^p \right)^{\frac{1}{p}}, p \geq 1$. $p = \infty$ essentially corresponds to $\|x - y\|_\infty$.

Problem 1. Show that \mathbb{L}_p is not a metric when $0 < p < 1$.

Another important example is $S = C[0, T]$ – the space of continuous functions $x : [0, T] \rightarrow \mathbb{R}^d$ and $\rho(x, y) = \rho_T = \sup_{0 \leq t \leq T} \|x(t) - y(t)\|$, where $\|\cdot\|$ can be taken as any of \mathbb{L}_p or \mathbb{L}_∞ . We will usually concentrate on the case $d = 1$, in which case $\rho(x, y) = \rho_T = \sup_{0 \leq t \leq T} |x(t) - y(t)|$. The space $C[0, \infty)$ is

also a metric space under $\rho(x, y) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} \min(\rho_n(x, y), 1)$, where ρ_n is the metric defined on $C[0, n]$. (Why did we have to use the min operator in the definition above?). We call ρ_T and ρ uniform metric. We will also write $\|x - y\|_T$ or $\|x - y\|$ instead of ρ_T .

Problem 2. Establish that ρ_T and ρ defined above on $C[0, \infty)$ are metrics. Prove also that if $x_n \rightarrow x$ in ρ metric for $x_n, x \in C[0, \infty)$, then the restrictions x'_n, x' of x_n, x onto $[0, T]$ satisfy $x'_n \rightarrow x'$ w.r.t. ρ_T .

Finally, let us give an example of a metric space from a graph theory. Let $G = (V, E)$ be an undirected graph on nodes V and edges E . Namely, each element (edge) of E is a pair of nodes (u, v) , $u, v \in V$. For every two nodes u and v , which are not necessarily connected by an edge, let $\rho(u, v)$ be the length of a shortest path connecting u with v . Then it is easy to see that ρ is a metric on the finite set V .

Definition 2. A sequence $x_n \in S$ is said to converge to a limit $x \in S$ (we write $x_n \rightarrow x$) if $\lim_n \rho(x_n, x) = 0$. A sequence $x_n \in S$ is Cauchy if for every $\epsilon > 0$ there exists n_0 such that for all $n, n' > n_0$, $\rho(x_n, x_{n'}) < \epsilon$. A metric space is defined to be complete if every Cauchy sequence converges to some limit x .

The space \mathbb{R}^d is a complete space under all three metrics $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_\infty$. The space \mathbb{Q} of rational points in \mathbb{R} is not complete (why?). A subset $A \subset S$ is called dense if for every $x \in S$ there exists a sequence of points $x_n \in A$ such that $x_n \rightarrow x$. The set of rational values in \mathbb{R} is dense and is countable. The set of irrational points in \mathbb{R} is dense but not countable. The set of points $(q_1, \dots, q_d) \in \mathbb{R}^d$ such that q_i is rational for all $1 \leq i \leq d$ is a countable dense subset of \mathbb{R}^d .

Definition 3. A metric space is defined to be separable if it contains a dense countable subset A . A metric space S is defined to be a **Polish** space if it is complete and separable.

We just realized that \mathbb{R}^d is Polish. Is there a countable dense subset of $C[0, T]$ of $C[0, \infty)$, namely are these spaces Polish as well? The answer is yes, but we will get to this later.

Problem 3. Given a set S , consider the metric ρ defined by $\rho(x, x) = 0$, $\rho(x, y) = 1$ for $x \neq y$. Show that (S, ρ) is a metric space. Suppose S is uncountable. Show that S is not separable.

Given $x \in S$ and $r > 0$ define a ball with radius r to be $B(x, r) = \{y \in S : \rho(x, y) \leq r\}$. A set $A \subset S$ is defined to be open if for every $x \in A$ there exists

ϵ such that $B(x, \epsilon) \subset A$. A set A is defined to be closed if $A^c = S \setminus A$ is open. The empty set is assumed to be open and closed by definition. It is easy to check that union of open sets is open and intersection of closed sets is closed. Also it is easy to check that a finite intersection of open sets is open and finite union of closed sets is closed. Every interval (a, b) is and every countable union of intervals $\cup_{i \geq 1} (a_i, b_i)$ is an open set. Show that a union of close intervals $[a_i, b_i], i \geq 1$ is not necessarily close.

For every set A define its interior A° as the union of all open sets $U \subset A$. This set is open (check). For every set A define its closure \bar{A} as the intersection of all closed sets $V \supset A$. This set is closed. For every set A define its boundary ∂A as $\bar{A} \setminus A^\circ$. Examples of open sets are open balls $B_o(x, r) = \{y \in S : \rho(x, y) < r\} \subset B(x, r)$ (check this). A set $K \subset S$ is defined to be *compact* if every sequence $x_n \in K$ contains a converging subsequence $x_{n_k} \rightarrow x$ and $x \in K$. It can be shown that $K \subset \mathbb{R}^d$ is compact if and only if K is closed and bounded (namely $\sup_{x \in K} \|x\| < \infty$ (this applies to any \mathbb{L}_p metric). Prove that every compact set is closed.

Proposition 1. *Given a metric space (S, ρ) a set K is compact iff every cover of K by open sets contains a finite subcover. Namely, if $U_r, r \in R$ is a (possibly uncountable) family of sets such that $K \subset \cup_r U_r$, then there exists a finite subset $r_1, \dots, r_m \in R$ such that $K \subset \cup_{1 \leq i \leq m} U_{r_i}$.*

We skip the proof of this fact, but it can be found in any book on topology.

Problem 4. *Give an example of a closed bounded set $K \subset C[0, T]$ which is not compact.*

Definition 4. *Given two metric spaces $(S_1, \rho_1), (S_2, \rho_2)$ a mapping $f : S_1 \rightarrow S_2$ is defined to be continuous in $x \in S_1$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Equivalently for every y such that $\rho_1(x, y) < \delta$ we must have $\rho_2(f(x), f(y)) < \epsilon$. And again, equivalently, if for every sequence $x_n \in S_1$ converging to $x \in S$ it is also true that $f(x_n)$ converges to $f(x)$.*

A mapping f is defined to be continuous if it is continuous in every $x \in S_1$. A mapping is uniformly continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\rho_1(x, y) < \delta$ implies $\rho_2(f(x), f(y)) < \epsilon$.

Problem 5. *Show that f is a continuous mapping if and only if for every open set $U \subset S_2, f^{-1}(U)$ is an open set in S_1 .*

Proposition 2. *Suppose $K \subset S_1$ is compact. If $f : K \rightarrow \mathbb{R}^d$ is continuous then it is also uniformly continuous. Also there exists $x_0 \in K$ satisfying $\|f(x_0)\| = \sup_{x \in K} \|f(x)\|$, for any norm $\|\cdot\| = \|\cdot\|_p$.*

Proof. This is where alternative property of compactness provided in Proposition 1 is useful. Fix $\epsilon > 0$. For every $x \in K$ find $\delta = \delta(x)$ such that $f(B(x, \delta(x))) \subset B(f(x), \epsilon)$. This is possible by continuity of f . Then $K \subset \cup_{x \in K} B_o(x, \delta(x)/2)$ (recall that B_o is an "open" version of B). Namely, we have an open cover of K . By Proposition 1, there exists a finite subcover $K \subset \cup_{1 \leq i \leq k} B_o(x_i, \delta(x_i)/2)$. Let $\delta = \min_{1 \leq i \leq k} \delta(x_i)$. This value is positive since k is finite. Consider any two points $y, z \in K$ such that $\rho_1(y, z) < \delta/2$. We just showed that there exists $i, 1 \leq i \leq k$ such that $\rho_1(x_i, y) \leq \delta(x_i)/2$. By triangle inequality $\rho_1(x_i, z) < \delta(x_i)/2 + \delta/2 \leq \delta(x_i)$. Namely both y and z belong to $B_o(x_i, \delta(x_i))$. Then $f(y), f(z) \in B_o(f(x_i), \epsilon)$. By triangle inequality we have $\|f(y) - f(z)\| \leq \|f(y) - f(x_i)\| + \|f(z) - f(x_i)\| < 2\epsilon$. We conclude that for every two points y, z such that $\rho_1(y, z) < \delta/2$ we have $\|f(y) - f(z)\| < 2\epsilon$. The uniform continuity is established. Notice, that in this proof the only property of the target space \mathbb{R}^d we used is that it is a metric space. In fact, this part of the proposition is true if \mathbb{R}^d is replaced by any metric space S_2, ρ_2 .

Now let us show the existence of $x_0 \in K$ satisfying $\|f(x_0)\| = \sup_{x \in K} \|f(x)\|$. First let us show that $\sup_{x \in K} \|f(x)\| < \infty$. If this is not true, identify a sequence $x_n \in K$ such that $\|f(x_n)\| \rightarrow \infty$. Since K is compact, there exists a subsequence x_{n_k} which converges to some point $y \in K$. Since f is continuous then $f(x_{n_k}) \rightarrow f(y)$, but this contradicts $\|f(x_n)\| \rightarrow \infty$. Thus $\sup_{x \in K} \|f(x)\| < \infty$. Find a sequence x_n satisfying $\lim_n \|f(x_n)\| = \sup_{x \in K} \|f(x)\|$. Since K is compact there exists a converging subsequence $x_{n_k} \rightarrow x_0$. Again using continuity of f we have $f(x_{n_k}) \rightarrow f(x_0)$. But $\|f(x_{n_k})\| \rightarrow \sup_{x \in K} \|f(x)\|$. We conclude $f(x_0) = \sup_{x \in K} \|f(x)\|$. \square

We mentioned that the sets in \mathbb{R}^d which are compact are exactly bounded closed sets. What about $C[0, T]$? We will need a characterization of compact sets in this space later when we analyze tightness properties and construction of a Brownian motion.

Given $x \in C[0, T]$ and $\delta > 0$, define $w_x(\delta) = \sup_{s, t: |s-t| < \delta} |x(t) - x(s)|$. The quantity $w_x(\delta)$ is called *modulus of continuity*. Since $[0, T]$ is compact, then by Proposition 2 every $x \in C[0, T]$ is uniformly continuous on $[0, T]$. This may be restated as for $\epsilon > 0$, there exists $\delta > 0$ such that $w_x(\delta) < \epsilon$.

Theorem 1 (Arzelá-Ascoli Theorem). *A set $A \subset C[0, T]$ is compact if and*

only if it is closed and

$$\sup_{x \in A} |x(0)| < \infty, \quad (1)$$

and

$$\limsup_{\delta \rightarrow 0} \sup_{x \in A} w_x(\delta) = 0. \quad (2)$$

Proof. We only show that if A is compact then (1) and (2) hold. The converse is established using a similar type of mathematical analysis/topology arguments.

We already know that if A is compact it needs to be closed. The assertion (1) follows from Proposition 2. We now show (2). For any $s, t \in [0, T]$ we have

$$|y(t) - y(s)| \leq |y(t) - x(t)| + |x(t) - x(s)| + |x(s) - y(s)| \leq |x(t) - x(s)| + 2\|x - y\|.$$

Similarly we show that $|x(t) - x(s)| \leq |y(t) - y(s)| + 2\|x - y\|$. Therefore for every $\delta > 0$.

$$|w_x(\delta) - w_y(\delta)| < 2\|x - y\|. \quad (3)$$

We now show (2). Check that (2) is equivalent to

$$\limsup_n \sup_{x \in A} w_x\left(\frac{1}{n}\right) = 0. \quad (4)$$

Suppose A is compact but (4) does not hold. Then we can find a subsequence $x_{n_i} \in A, i \geq 1$ such that $w_{x_{n_i}}(1/n_i) \geq c$ for some $c > 0$. Since A is compact then there is further subsequence of x_{n_i} which converges to some $x \in A$. To ease the notation we denote this subsequence again by x_{n_i} . Thus $\|x_{n_i} - x\| \rightarrow 0$. From (3) we obtain

$$|w_x(1/n_i) - w_{x_{n_i}}(1/n_i)| < 2\|x - x_{n_i}\| < c/2$$

for all i larger than some i_0 . This implies that

$$w_x(1/n_i) \geq c/2, \quad (5)$$

for all sufficiently large i . But x is continuous on $[0, T]$, which implies it is uniformly continuous, as $[0, T]$ is compact. This contradicts (5). □

2 Convergence of mappings

Given two metric spaces $(S_1, \rho_1), (S_2, \rho_2)$ a sequence of mappings $f_n : S_1 \rightarrow S_2$ is defined to be point-wise converging to $f : S_1 \rightarrow S_2$ if for every $x \in S_1$ we have $\rho_2(f_n(x), f(x)) \rightarrow 0$. A sequence f_n is defined to converge to f uniformly if

$$\lim_n \sup_{x \in S_1} \rho_2(f_n(x), f(x)) = 0.$$

Also given $K \subset S_1$, sequence f_n is said to converge to f uniformly on K if the restriction of f_n, f onto K gives a uniform convergence. A sequence f_n is said to converge to f *uniformly on compact sets u.o.c* if f_n converges uniformly to f on every compact set $K \subset S_1$.

Problem 6. Let $S_1 = [0, \infty)$ and let S_2 be arbitrary. Show that f_n converges to f uniformly on compact sets if and only if for every $T > 0$

$$\lim_n \sup_{0 \leq t \leq T} \rho_2(f_n(t), f(t)) = 0.$$

Point-wise convergence does not imply uniform convergence even on compact sets. Consider $x_n = nx$ for $x \in [0, 1/n]$, $= n(2/n - x)$ for $x \in [1/n, 2/n]$ and $= 0$ for $x \in [2/n, 1]$. Then x_n converges to zero function point-wise but not uniformly. Moreover, if f_n is continuous and f_n converges to f point-wise, this does not imply in general that f is continuous. Indeed, let $f_n = 1/(nx+1), x \in [0, 1]$. Then f_n converges to 0 point-wise everywhere except $x = 0$ where it converges to 1. The limiting function is discontinuous. However, the uniform continuity implies continuity of the limit, as we are about to show.

Proposition 3. Suppose $f_n : S_1 \rightarrow S_2$ is a sequence of continuous mappings which converges uniformly to f . Then f is continuous as well.

Proof. Fix $x \in S_1$ and $\epsilon > 0$. There exists n_0 such that for all $n > n_0$, $\sup_z \rho_2(f_n(z), f(z)) < \epsilon/3$. Fix any such $n > n_0$. Since, by assumption f_n is continuous, then there exists $\delta > 0$ such that $\rho_2(f_n(x), f_n(y)) < \epsilon/3$ for all $y \in B_o(x, \delta)$. Then for any such y we have

$$\rho_2(f(x), f(y)) \leq \rho_2(f(x), f_n(x)) + \rho_2(f_n(x), f_n(y)) + \rho_2(f_n(y), f(y)) < 3\epsilon/3 = \epsilon.$$

This proves continuity of f . □

Theorem 2. The spaces $C[0, T], C[0, \infty)$ are Polish.

Problem 7. Use Proposition 3 (or anything else useful) to prove that $C[0, T]$ is complete.

That $C[0, T]$ has a dense countable subset can be shown via approximations by polynomials with rational coefficients (we skip the details).

3 Skorohod space and Skorohod metric

The space $C[0, \infty)$ equipped with uniform metric will be convenient when we discuss Brownian motion and its application later on in the course, since Brownian motion has continuous samples. Many important processes in practice, however, including queueing processes, storage, manufacturing, supply chain, etc. are not continuous, due to discrete quantities involved. As a result we need to deal with probability concept on spaces of not necessarily continuous functions.

Denote by $D[0, \infty)$ the space of all functions x on $[0, \infty)$ taking values in \mathbb{R} or in general any metric space (S, ρ) , such that x is right-continuous and has left limits. Namely, for every t_0 , $\lim_{t \uparrow t_0} f(t)$, $\lim_{t \downarrow t_0} f(t)$ exist, and $\lim_{t \downarrow t_0} f(t) = f(t_0)$. As an example, think about a process describing the number of customers in a branch of a bank. This process is described as a piece-wise constant function. We adopt a convention that at a moment when a customer arrives/departs, the number of customers is identified with the number of customers *right after* arrival/departure. This makes the process right-continuous. It also has left-limits, since it is piece-wise constant.

Similarly, define $D[0, T]$ to be the space of right-continuous functions on $[0, T]$ with left limits. We will right shortly RCLL. On $D[0, T]$ and $D[0, \infty)$ we would like to define a metric which measures some proximity between the functions (processes). We can try to use the uniform metric again. Let us consider the following two processes $x, y \in D[0, T]$. Fix $\tau, \in [0, T]$ and $\delta > 0$ such that $\tau + \delta < T$ and define $x(z) = 1\{z \geq \tau\}, y(z) = 1\{z \geq \tau + \delta\}$. We see that x and y coincide everywhere except for a small interval $[\tau, \tau + \delta)$. It makes sense to assume that these processes are "close" to each other. Yet $\|x - y\|_T = 1$. Thus uniform metric is inadequate. For this reason Skorohod introduce the so called *Skorohod* metric. Before we define Skorohod metric let us discuss the idea behind it. The problem with uniform metric was that the two processes x, y described above were close to each other in a sense that one is a perturbed version of the other, where the amount of perturbation is δ . In particular, consider the following piece-wise linear function $\lambda : [0, T] \rightarrow [0, T]$ given by

$$\lambda(t) = \begin{cases} \frac{\tau}{\tau + \delta}t, & t \in [0, \tau + \delta); \\ \tau + \frac{1 - \tau}{1 - \tau - \delta}(t - \tau - \delta), & t \in [\tau + \delta, T]. \end{cases}$$

We see that $x(\lambda(t)) = y(t)$. In other words, we rescaled the axis $[0, T]$ by a small amount and made y close to (in fact identical to) x . This motivates the following definition. From here on we use the following notations: $x \wedge y$ stands for $\min(x, y)$ and $x \vee y$ stands for $\max(x, y)$

Definition 5. Let Λ be the space of strictly increasing continuous functions λ from $[0, T]$ onto $[0, T]$. A Skorohod metric on $D[0, T]$ is defined by

$$\rho_s(x, y) = \inf_{\lambda \in \Lambda} \left(\|\lambda - I\| \vee \|x - y\lambda\| \right),$$

for all $x, y \in D[0, T]$, where $I \in \Lambda$ is the identity transformation, and $\|\cdot\|$ is the uniform metric on $D[0, T]$.

Thus, per this definition, the distance between x and y is less than ϵ if there exists $\lambda \in \Lambda$ such that $\sup_{0 \leq t \leq T} |\lambda(t) - t| < \epsilon$ and $\sup_{0 \leq t \leq T} |x(t) - y(\lambda(t))| < \epsilon$.

Problem 8. Establish that ρ_s is a metric on $D[0, T]$.

Proposition 4. The Skorohod metric and uniform metric are equivalent on $C[0, T]$, in a sense that for $x_n, x \in C[0, T]$, the convergence $x_n \rightarrow x$ holds under Skorohod metric if and only if it holds under the uniform metric.

Proof. Clearly $\|x - y\| \geq \rho_s(x, y)$. So convergence under uniform metric implies convergence under Skorohod metric. Suppose now $\rho_s(x_n, x) \rightarrow 0$. We need to show $\|x_n - x\| \rightarrow 0$.

Consider any sequence $\lambda_n \in \Lambda$ such that $\|\lambda_n - I\| \rightarrow 0$ and $\|x(\lambda_n) - x_n\| \rightarrow 0$. Such a sequence exists since $\rho_s(x_n, x) \rightarrow 0$ (check). We have

$$\|x - x_n\| \leq \|x - x\lambda_n\| + \|x\lambda_n - x_n\|.$$

The second summand in the right-hand side converges to zero by the choice of λ_n . Also since λ_n converges to I uniformly, and x is continuous on $[0, T]$ and therefore uniformly continuous on $[0, T]$, then $\|x - x\lambda_n\| \rightarrow 0$. \square

4 Additional reading materials

- Billingsley [1], Appendix M1-M10.

References

- [1] P. Billingsley, *Convergence of probability measures*, Wiley-Interscience publication, 1999.

MIT OpenCourseWare
<http://ocw.mit.edu>

15.070J / 6.265J Advanced Stochastic Processes
Fall 2013

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.