METRIC SPACES

TOPIC: BOUNDARY

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Chapter: Boundary

" *The moving power of mathematical invention*

is not reasoning but imagination."

Augustus De Morgan, 1806– 1871.

1.LEARNING OUTCOMES

With this chapter the reader gets exposed to notion of boundary point and also to notions like closure, interior, exterior etc., that arise from the concept of boundary. Further will see how these concepts are related to other concepts like isolated points, accumulation points etc. A detailed treatment of the notion of boundary point is given in this chapter and thereby preparing the reader for further topics.

2. PREREQUISITES

This chapter requires the reader to have familiarity with definition of metric space and all the standard examples of metric spaces. Also, the reader should be familiar with the notion of diameter of any given set, distance of a point from a given set, isolated points and accumulation points.

3. PRELIMINARIES

To make the text self-contained we state the some definitions and results that we explicitly use in this chapter.

Definition 3.1 Let X be a non-empty set and let

$$d : X \times X \to \mathbb{R}^{+} \cup \{0\}$$

be a non-negative real valued function satisfying the following conditions :

[M1] $d(x,y) = 0 \iff x = y$;

[M2] $d(x,y) = d(y,x)$, for all x, y in X ; **[Symmetry Property]**

[M3] If x, y, z are any three elements of X, then **[Triangle Inequality]**

$$d(x,z) \leq d(x,y) + d(y,z).$$

Then d is called a **metric** on X and (X,d) is called a **metric space**. For each x, y in X, we call the number $d(x,y)$ the **distance** between x and y w.r.t the metric d.

Definition 3.2 Let (X,d) be a metric space and A be any subset of X. Then the diameter of set A, denoted by $diam(A)$, is defined to be

$$diam(A) = \sup\{\, d(u,v) : u, v \in A\}.$$

Theorem 3.3 For any subset S of $\mathbb{R}$, $diam(S) = \sup S - \inf S$.

Definition 3.4 Let (X,d) be a metric space, A be a subset of X and x be any point in X. The distance from x to A, denoted by $dist(x,A)$, is defined as

$$dist(x,A) = \inf\{d(x,a) : x \in A\}.$$

By definition of infimum, for each $\epsilon \in \mathbb{R}^{+}$, there exists $w \in A$ such that

$$\boldsymbol{d(x,w) < dist\,(x,A) + \epsilon}.$$

Theorem 3.5 Suppose X is a metric space, $x \in X$ and A and B are non-empty subsets of X for which $A \subseteq B$. Then

$$dist(x,B) \le dist(x,A) \le dist(x,B) + diam(B) .$$

Theorem 3.6 *For any x in $\mathbb{R}$, $\boldsymbol{dist(x,\mathbb{Q}) = 0}$ **and** $\boldsymbol{dist(x,\mathbb{R}/\mathbb{Q}) = 0}$.*

Definition 3.7 Let (X,d) be a metric space, S be a subset of X and $z \in S$. Then z is called an **isolated point** of S if distance of z from the rest of the points of S is non-zero i.e.,

$$dist(z, S\backslash\{z\}) \neq 0.$$

In this case, we say that z is isolated in S. The collection of all isolated points of S will be denoted by $iso(S)$.

Theorem 3.8 Suppose (X,d) is a metric space and $A \subseteq B \subseteq X$. Then $A \cap iso(B) \subseteq iso(A)$.

Definition 3.9 Suppose X is a metric space, $z \in X$ and S is a subset of X. Then z is called an **accumulation point** or a **limit point** of S in X if $dist(z, S\backslash\{z\}) = 0$. The collection of all accumulation points of S in X will be denoted by $acc(S)$ or by $acc_X(S)$ if there is necessity of specifying the space under consideration.

Theorem 3.10 Suppose (X,d) is a metric space and $A \subseteq B \subseteq X$. Then $acc(A) \subseteq acc(B)$.

Theorem 3.11 Suppose (X,d) is a metric space and $A \subseteq B \subseteq X$. Then

$$acc_B(A) = B \cap acc_X(A).$$

4. BOUNDARY

The dictionary meaning of boundary is ''the line or plane indicating the limit or extent of something''. Another way to look at the boundary is that it forms the connection between the things inside it and the things outside. In other words if we remove the boundary then the connection (distinction) between the inner and outer world will be lost. In Mathematics also, we intend to correspond with this intuitive idea while defining the notion of boundary in metric spaces. In fact this intuitive idea of connection through boundary forms the motive of defining notion of connectedness in metric spaces through boundary. These points will become more clear once we will deal with some examples.

4.1 BOUNDARY POINTS

Given a metric space (X,d) and a subset S of X, we already know that each point of X is either in S or in S^c. Also, each point in S is at zero distance from S and each point in S^c is at zero distance from S^c. With boundary points of S we mean only those special points of X that are at zero distance both from S and from S^c. We now look at its formal definition.

Definition 4.1.1 Let (X,d) be a metric space, S be any subset of X and a be any point in X. Then a is called a **boundary point** of S in X if

$$dist(a, S) = 0 = dist(a, S^c).$$

The collection of all boundary points of S in X is called the **boundary** of S in X and is denoted by ∂S or $\partial_X S$ if there is need to specify the metric space under consideration is X.

> **Given any metric space (X, d), the distance of each point in the space from the empty set is ∞, therefore**
> $$\partial\emptyset = \emptyset \ \text{ and } \ \partial X = \emptyset\,.$$

Theorem 4.1.2 Let(X, d)be a metric space and S be a subset of X. Then $\partial S = \partial(S^c)$.

Proof: Consider

$$x \in \partial S$$
$$\Leftrightarrow \ dist(a, S) = 0 = dist(a, S^c)$$
$$\Leftrightarrow \ dist(a, (S^c)^c) = 0 = dist(a, S^c)[\because \ (S^c)^c = S]$$
$$\Leftrightarrow \ x \in \partial(S^c)$$

Hence the equality follows. ■

> **Let (X, d) be a metric space and $S \subseteq X$. Then**
> $$\partial S = \partial(S^c)\,.$$

Example 4.1.3 On the real line $(\mathbb{R}, u)$ with usual metric u, $\boldsymbol{\partial(a, b) = \{a, b\}}$**for any** $\boldsymbol{a, b} \in \mathbb{R}$.

Claim 1.$a$is a boundary point of (a, b).

For any positive real number ϵ $(0 < \epsilon < b - a)$, it is clear that $a + \epsilon \in (a, b)$. Thus

$$dist\big(a, (a, b)\big) \le d(a, a + \epsilon) = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get $dist\big(a, (a, b)\big) = 0$. Also

$$dist(a, (a, b)^c) = 0\,. \qquad [\because \ a \in (a, b)^c]$$

It follows that $a \in \partial(a, b)$.

Claim 2.$b$is a boundary point of (a, b) i.e., $b \in \partial(a, b)$.

For any positive real number ϵ $(0 < \epsilon < b - a)$, $b - \epsilon \in (a, b)$ and therefore

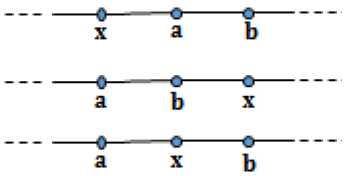
$$dist\big(b, (a, b)\big) \le d(b, b - \epsilon) = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get $dist\big(b, (a, b)\big) = 0$. Also,

$$dist(b, (a, b)^c) = 0. \qquad [\because \ b \in (a, b)^c]$$

Hence $b \in \partial(a, b)$.

Claim 3.$x \notin \partial(a, b) \quad \forall \ x \in \mathbb{R}\backslash\{a, b\}$.



Consider any $x \in \mathbb{R}\backslash\{a, b\}$. If $x \notin [a, b]$, then by definition

$$dist\big(x, (a, b)\big) = \min\{|x - a|\,, |x - b|\} > 0$$
$$\Rightarrow \quad x \notin \partial(a, b)\,.$$

If $x \in (a, b)$, then

$$dist(x, (a, b)^c) = \min\{\,x - a, b - x\,\} > 0$$
$$\Rightarrow \quad x \notin \partial(a, b)\,.$$

Hence $x \notin \partial(a,b) \quad \forall\ x \in \mathbb{R}\backslash\{a,b\}$.

Thus from all the three claims it follows that

$$\partial(a,b) = \{a,b\}. \qquad \blacksquare$$

Example 4.1.4 On the real line $(\mathbb{R}, u)$, similar to Example 4.1.3, we can show that for any $a, b \in \mathbb{R}$,

$$\partial[a,b) = \partial(a,b] = \partial[a,b] = \{a,b\}$$
$$\partial(a,\infty) = \partial[a,\infty) = \partial[a,a] = \{a\}$$
$$\partial(-\infty,b) = \partial(-\infty,b] = \{b\}$$
$$\partial(-\infty,\infty) = \emptyset\,. \qquad \blacksquare$$

Example 4.1.5 Consider the complex plane $(\mathbb{C}, d)$, where d is the usual metric on $\mathbb{C}$ and the closed unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ of the complex plane. The unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the set of all boundary points of $\mathbb{D}$i.e., $\partial\mathbb{D} = \mathbb{T}$.

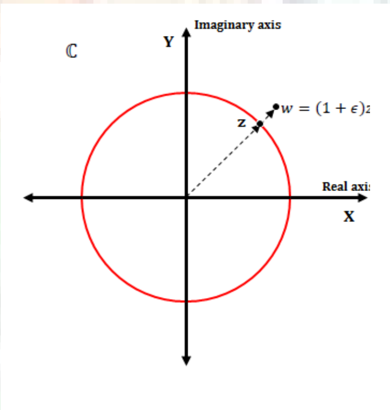
Proof: Let $z$be any point in $(i.e., |z| = 1)$. Since $z \in \mathbb{D}$, $dist(z,\mathbb{D}) = 0$.Thus to prove $\mathbb{T} \subseteq \partial\mathbb{D}$, it is enough to show that $dist(z,\mathbb{D}^c) = 0$. Let $\epsilon > 0$ be any real number. Clearly,

$$w = (1+\epsilon)z \in \mathbb{D}^c. \qquad [\because |w| = |(1+\epsilon)z| = (1+\epsilon)|z| = 1+\epsilon > 1]$$

Now

$$dist(z,\mathbb{D}^c) \leq d(z,w) = |z-w| = |\epsilon z| = \epsilon|z| = \epsilon.$$

Since ϵ is arbitrary, we get $dist(z,\mathbb{D}^c) = 0$ .Hence $\mathbb{T} \subseteq \partial\mathbb{D}$.



Next we claim that$\partial\mathbb{D} \subseteq \mathbb{T}$ i.e.,$\partial\mathbb{D} \cap (\mathbb{C}\backslash\mathbb{T}) = \emptyset$. Consider any $w \in \mathbb{C}\backslash\mathbb{T}$.

Case 1.$w \in \mathbb{D}$ i.e, $|w| < 1$.

For $\epsilon = 1 - |w| > 0$, there exists $u \in \mathbb{D}^c$ such that $d(w,u) < dist(w,\mathbb{D}^c) + \epsilon$. Thus

$$\begin{aligned} dist(w,\mathbb{D}^c) &> d(w,u) - \epsilon \\ &= |u-w| - \epsilon \\ &\geq |u| - |w| - (1-|w|) \\ &= |u| - 1 \ > 0. \qquad [\because u \in \mathbb{D}^c] \end{aligned}$$

Therefore it follows that $dist(w,\mathbb{D}^c) > 0$ and hence $w \notin \partial\mathbb{D}$.

Case 2.$w \notin \mathbb{D}$ i.e, $|w| > 1$.

For $\epsilon = |w| - 1 > 0$, there exists $u \in \mathbb{D}$ such that $d(w,u) < dist(w,\mathbb{D}) + \epsilon$. Thus

$$\begin{aligned} dist(w,\mathbb{D}) &> d(w,u) - \epsilon \\ &= |u-w| - \epsilon \\ &\geq |w| - |u| - (|w|-1) \\ &= 1 - |u| > 0. \end{aligned}$$

It follows that $dist(w,\mathbb{D}) > 0$ and hence $w \notin \partial\mathbb{D}$.

Thus from both the cases $\cap\, \mathbb{C}\backslash\mathbb{T} = \emptyset$. Hence$\mathbb{T} = \partial\mathbb{D}$. $\blacksquare$

On similar lines it can be shown that the points of $\mathbb{T}$ are the boundary points of the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$. In fact, given any disc $\mathbb{A}$ of $\mathbb{C}$ such that

$$\{z \in \mathbb{C} : |z| < 1\} \subseteq \mathbb{A} \subseteq \mathbb{D}$$

the points in $\mathbb{T}$ are boundary points of disc $\mathbb{A}$ i.e., $\partial\mathbb{A} = \mathbb{T}$.

So far our intuitive idea of a boundary of a set and the actual boundary of that set are coinciding with each other. For example on the real line $\mathbb{R}$, intuitively we know that boundary of interval (a,b) should be the set$\{a,b\}$ and this in fact is correct. Also, in case of complex plane, we have seen that intuitive boundary of the closed unit disc and its actual boundary coincides. Naturally, one may think that intuitive idea about boundary points is going to be correct all the time. But there are examples where we will have completely different boundary from what our intuition says. Next two examples will make this point more clear.

Example 4.1.6 Consider the interval $(7,17]$ in the complex plane $\mathbb{C}$. As in case of real line $\mathbb{R}$, we may think that 7 and 17 are the only boundary points of (7,17]. However, every point of the interval $[7,17]$ is a boundary point of $(7,17]$ in the complex plane $\mathbb{C}$.

Proof: Let $z \in \mathbb{C}$ be any point and $S = (7,17]$.

<u>Case 1.</u>$z \in [7,17]$ i.e.,$z = x + i0$ $for\ some\ x \in [7,17]$

Clearly, $dist(z,S) = 0$. We claim that $dist(z,S^c) = dist(z,\mathbb{C}\backslash S) = 0$. Let ϵ be any positive real number. Obviously, $z' = x + i\epsilon \in S^c$ and hence $dist(z,S^c) \leq d(z,z') = \epsilon$. Since $\epsilon > 0$ is arbitrary, we get $dist(z,S^c) = 0$. Thus $z \in \partial S$.

<u>Case 2.</u>$z \notin [7,17]$

Let $z = a + ib$ for some $a,b \in \mathbb{R}$. We have two possibilities:

<u>Sub case 1.</u>$a \in [7,17]$

Then $b \neq 0$. For $\epsilon = |b|/2 > 0$, there exists $z' \in S$ such that $d(z,z') < dist(z,S) + \epsilon$. Then

$$\begin{aligned} dist(z,S) &> d(z,z') - \epsilon = |z - z'| - \epsilon \\ &= \sqrt{(a - z')^2 + b^2} - \frac{|b|}{2} \\ &\geq |b| - \frac{|b|}{2} = \frac{|b|}{2} > 0 . \end{aligned}$$

Thus $dist(z,S) > 0$ in this sub case.

<u>Sub case 2.</u>$a \notin [7,17]$

For

$$\epsilon = \frac{\min\{|a-7|, |a-17|\}}{2} > 0,$$

there exists a point $z' \in S$ such that $d(z,z') < dist(z,S) + \epsilon$. Then

$$\begin{aligned} dist(z,S) &> d(z,z') - \epsilon = |z - z'| - \epsilon \\ &= \sqrt{(a - z')^2 + b^2} - \epsilon \\ &\geq |a - z'| - \epsilon \end{aligned}$$

$$\geq 2\epsilon - \epsilon > 0. \qquad [\because |a - z'| \geq \min\{|a-7|, |a-17|\}]$$

Thus in this sub case also, $dist(z,S) > 0$.

Hence from both the sub cases $dist(z,S) > 0$ as claimed. And therefore $z \notin \partial S$.

Thus from both the cases it follows that $\partial S = [7,17]$. ∎

Example 4.1.7 Consider the metric space $X = [0,1] \cup [7,8]$ with the usual metric induced from the real line $\mathbb{R}$. Let $S = [0,1]$. There is a temptation to respond without thinking that 0 and 1 are boundary points of the set S. But we will see that boundary of S is empty. Consider any point x in X. If $x \in S$, then $dist(x, S^c) = dist(x, [7,8]) \geq 6$ and therefore $x \notin \partial S$. If $x \notin S$, then $x \in [7,8]$ so that $dist(x,S) \geq 6$ and hence $x \notin \partial S$. Thus in either case $x \notin \partial S$. Hence $\partial S = \emptyset$. ∎

An important lesson is to be learnt from these two examples:

Boundary of a set is always calculated relative to the metric space of which the set is considered as a subset.

We have seen that both accumulation points and boundary points are defined using zero distances. But accumulation points need not be boundary points. For example all the points in the disc $\{z \in \mathbb{C} : |z| < 1\}$ are accumulation points of the disc, and none of them is a boundary point. Thus it becomes important to study the relationship between accumulation points and boundary points.

Theorem 4.1.8 Let (X,d) be a metric space, S be a subset of X and $a \in X$.

(i). If $a \notin S$, then $a \in \partial S$ if, and only if, $a \in acc(S)$.

(ii). If $a \in S$, then $a \in \partial S$ if, and only if, $a \in acc(S^c)$.

Proof: (i) If $a \notin S$, then $S = S\backslash\{a\}$ and $dist(a, S^c) = 0$. Now

$$\begin{aligned} & a \in \partial S \\ \Leftrightarrow\ & dist(a,S) = 0 \qquad [\because dist(a,S^c) = 0] \\ \Leftrightarrow\ & dist(a, S\backslash\{a\}) = 0 \\ \Leftrightarrow\ & a \in acc(S) \end{aligned}$$

(ii) If $a \in S$, then $a \notin S^c$ and hence by part (i),

$$a \in \partial(S^c) = \partial S \Leftrightarrow a \in acc(S^c) .$$

Alternatively, if $a \in S$, then $S^c = S^c\backslash\{a\}$ and $dist(a,S) = 0$. Now

$$\begin{aligned} & a \in \partial S \\ \Leftrightarrow\ & dist(a,S^c) = 0 \qquad [\because dist(a,S) = 0] \\ \Leftrightarrow\ & dist(a, S^c\backslash\{a\}) = 0 \\ \Leftrightarrow\ & a \in acc(S^c) \end{aligned}$$

∎

$a \in S$	$\Rightarrow$	$a \in \partial S = \partial(S^c)$	$\Leftrightarrow$	$a \in acc(S)$
$a \notin S$	$\Rightarrow$	$a \in \partial S = \partial(S^c)$	$\Leftrightarrow$	$a \in acc(S^c)$

So far whatever examples we have considered, it may be noted that $\partial(\partial S) = \partial S$ for each set S we considered. This naturally raises a question: For any subset S of a metric space, do we always have $\partial(\partial S) = \partial S$? The answer is not necessarily and there is a counterexample for this.

Example 4.1.10 On the real line $(\mathbb{R}, u)$, for any $x \in \mathbb{R}$,

$$dist(x, \mathbb{Q}) = 0 = dist(x, \mathbb{R}\backslash\mathbb{Q}) \ .$$

Therefore $\partial\mathbb{Q} = \mathbb{R}$. Since$\mathbb{R}$ has empty boundary, it follows that

$$\partial(\partial\mathbb{Q}) = \emptyset \neq \partial\mathbb{Q}.$$

■

Thus there exists a metric space X and a subset S of X such that $\partial(\partial S) \neq \partial S$. Next obvious question to ask is whether boundary of boundary of a set and boundary of that set is related or they are totally different entities. We will show that they are related to each other. In fact we will show that for any subset S of a metric space we always have $\partial(\partial S) \subseteq \partial S$.

Theorem 4.1.11 Let (X, d) be a metric space and S be a subset of X. Then $\partial(\partial S) \subseteq \partial S$.

Proof: If $\partial(\partial S) = \emptyset$, then we have nothing to prove. Let$\partial(\partial S) \neq \emptyset$and $x \in \partial(\partial S)$ be any arbitrary element.Then $dist(x, \partial S) = 0$ and therefore for any $\epsilon > 0$, there exists$z \in \partial S$ such that $d(x, z) < dist(x, \partial S) + \epsilon/2 = \epsilon/2$. Since $z \in \partial S$, therefore $dist(z, S) = 0 = dist(z, S^c)$. Hence for the above ϵ, there exists $p \in S$ and $q \in S^c$ such that

$$d(z, p) < dist(z, S) + \epsilon/2 = \epsilon/2 \quad and$$
$$d(z, q) < dist(z, S^c) + \epsilon/2 = \epsilon/2.$$

Thus

$$d(x, p) \leq d(x, z) + d(z, p) < \epsilon/2 + \epsilon/2 = \epsilon\,.$$

Since $p \in S$, therefore

$$dist(x, S) = \inf\{d(x, s) \ : \ s \in S\} \leq d(x, p) < \epsilon\,.$$

Similarly, we can obtain $dist(x, S^c) < \epsilon$. Since $\epsilon > 0$ is arbitrary, we get

$$dist(x, S) = 0 = dist(x, S^c).$$

Hence $x \in \partial S$and consequently,$\partial(\partial S) \subseteq \partial S$. ■

4.2 CLOSURE AND INTERIOR

As the name suggest, interior of a set is obtained from the set by removing all of its boundary points and closure of a set is obtained by adding to the set all of its boundary points. Since the concept of interior and closure are defined using boundary and boundary depends on the metric considered, therefore interior and closure also depends on the metric spaceconsidered.

Definition 4.2.1 Let (X,d) be a metric space, S be a subset of X. Then the **interior** of S in X is defined to be the difference $S\setminus\partial S$. We denote the interior of S in X by S^0or $Int(S)$. If it is necessary to specify the metric space, then we may write $Int_X(S)$. The members of the interior of S are called **interior points**of S.

Definition 4.2.2 Let (X,d) be a metric space, S be a subset of X. Then the **closure** of S in X is defined to be the union $S\cup\partial S$. We denote the closure of S in X by $\bar{S}$or $Cl(S)$. If it is necessary to specify the metric space, then we may write $Cl_X(S)$. The members of the closure of S are called **adherent points**of S.

Definition 4.2.3 Let (X,d) be a metric space, S be a subset of X. Then the **exterior** of S in X is defined to be the complement of the closure of Sin X. We denote the exterior of S in $X$by $Ext(S)$. If there is necessity of specifying the metric space, then we may write $Ext_X(S)$. The members of the exterior of S are called **exterior points**of S.

As known for any metric space X, boundary of both X and $\emptyset$ is $\emptyset$ i.e., $\partial X=\emptyset=\partial\emptyset$. Therefore

$$Cl(X) = X\cup\partial X = X\cup\phi = X \quad and$$
$$Int(X) = X\setminus\partial X = X\setminus\phi = X\,.$$

Similarly,$Cl(\emptyset)=\emptyset=Int(\emptyset)$.

Example 4.2.4 On the real line $\mathbb{R}$, we already know that for any x in $\mathbb{R}$

$$dist(x,\mathbb{Q}) = 0 = dist(x,\mathbb{R}\setminus\mathbb{Q})$$

Therefore the boundaries of $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$are $\mathbb{R}$. Hence it follows that $\overline{\mathbb{Q}}=\mathbb{R}=\overline{\mathbb{R}\setminus\mathbb{Q}}$.

Example 4.2.5 Since for the closed unit disc $\mathbb{D}=\{z\in\mathbb{C}: |z|\le 1\}$ and open unit disc $\mathbb{U}=\{z\in\mathbb{C}: |z|<1\}$,

$$\partial\mathbb{D}=\partial\mathbb{U}=\mathbb{T} \qquad [Theorem\ 4.1.5]$$

where$\mathbb{T}=\{z\in\mathbb{C}: |z|=1\}$ is the unit circle in the complex plane. It follows that

$$\overline{\mathbb{D}}=\mathbb{D}\cup\partial\mathbb{D}=\mathbb{D} \qquad and \qquad \mathbb{D}^0=\mathbb{D}\setminus\partial\mathbb{D}=\mathbb{D}\setminus\mathbb{T}=\mathbb{U}\,;$$
$$\overline{\mathbb{U}}=\mathbb{U}\cup\partial\mathbb{U}=\mathbb{D} \qquad and \qquad \mathbb{U}^0=\mathbb{U}\setminus\partial\mathbb{U}=\mathbb{U}\setminus\mathbb{T}=\mathbb{U}\,.$$

Example 4.2.6 On the real line $(\mathbb{R},u)$, from examples 4.1.3 and 4.1.4,for any $a,b\in\mathbb{R}$ with $a<b$,

$$\partial[a,b]=\partial(a,b)=\partial(a,b]=\partial[a,b)=\{a,b\};$$
$$\partial[a,\infty)=\partial(a,\infty)=\partial[a,a]=\{a\}$$
$$\partial(-\infty,b]=\partial(-\infty,b)=\{b\}$$
$$\partial(-\infty,\infty)=\emptyset.$$

Therefore

$$\overline{[a,b]}=\overline{(a,b)}=\overline{(a,b]}=\overline{[a,b)}=[a,b];$$
$$\overline{[a,\infty)}=\overline{(a,\infty)}=$$

$$[a,\infty)\overline{(-\infty,b]} = \overline{(-\infty,b)} = (-\infty,b]$$
$$\overline{[a,a]} = [a,a] = \{a\}$$
$$\overline{(-\infty,\infty)} = (-\infty,\infty).$$

Also,

$$[a,b]^0 = (a,b)^0 = (a,b]^0 = [a,b)^0 = (a,b);$$
$$[a,\infty)^0 = (a,\infty)^0 = (a,\infty)$$
$$(-\infty,b]^0 = (-\infty,b)^0 = (-\infty,b)$$
$$[a,a]^0 = \emptyset$$
$$(-\infty,\infty)^0 = (-\infty,\infty).$$ ■

Consider for any metric space (X,d) and any subset S of X

$$\overline{\partial S} = \partial S \cup \partial(\partial S) = \partial S. \qquad [\because \partial(\partial S) \subseteq \partial S \text{ (Theorem 4.1.11)}]$$

Also, $\partial S^0 = \partial S \backslash \partial(\partial S)$ and as already noticed $\partial(\partial S)$mayproperly be contained in ∂S. Therefore we may have non empty interior of a boundary. In fact the interior of boundary of $\mathbb{Q}$ in $\mathbb{R}$ is $\mathbb{R}$ itself i.e.,

$$Int\,(\partial\mathbb{Q}) = Int\,(\mathbb{R}) = \mathbb{R}.$$ ■

Next we have results on interior and closure and how they are related to accumulation points, isolated points, diameter etc.

Theorem 4.2.7 Let (X,d) be a metric space and S be a subset of X. Then

(i). $\bar{S} = S \cup acc(S)$;

(ii). $S^0 = S \backslash acc(S^c)$

Proof: From Theorem 4.1.8

$$\partial S \cap S^c = acc(S) \cap S^c \quad and$$
$$\partial S \cap S = acc(S^c) \cap S.$$

Now

$$\bar{S} = S \cup \partial S = S \cup (\partial S \cap S^c) = S \cup (acc(S) \cap S^c) = S \cup acc(S) \quad and$$
$$S^0 = S \backslash \partial S = S \backslash (\partial S \cap S) = S \backslash (acc(S^c) \cap S) = S \backslash acc(S^c).$$

Thus (i) and (ii) follows. ■

Theorem 4.2.8 Let (X,d) be a metric space and S be a subset of X. Then

(i). $(S^0)^c = \overline{S^c}$;

(ii). $(\bar{S})^c = (S^c)^0$ i.e., the exterior of S in X is the interior of its complement in X.

Proof:(i)$(S^0)^c = (S \setminus \partial S)^c$

$$= (S \cap (\partial S)^c)^c$$
$$= S^c \cup ((\partial S)^c)^c [Using\ De\ Morgan's\ Law]$$
$$= S^c \cup \partial S$$
$$= S^c \cup \partial(S^c) [Since\ \partial S = \partial(S^c), Theorem\ 4.1.2]$$
$$= \overline{S^c}$$

(ii)$(\bar{S})^c = (S \cup \partial S)^c$

$$= S^c \cap (\partial S)^c [Using\ De\ Morgan's\ Law]$$

$$= S^c \setminus \partial S$$

$$= S^c \setminus \partial(S^c) [Since\ \partial S = \partial(S^c), Theorem\ 4.1.2]$$

$$= (S^c)^0$$

Hence the theorem holds. ■

Theorem 4.2.9 Let (X,d) be a metric space and S be a subset of X. Then

(i). $\bar{S} = \{x \in X : dist(x,S) = 0\}$;

(ii). $Ext(S) = \{x \in X : dist(x,S) > 0\}$

(iii). $S^0 = \{x \in X : dist(x,S^c) > 0\}$.

Proof:(i) Let $x \in \bar{S}$, then either $x \in S$ or $x \in \partial S$, and in either case $dist(x,S) = 0$. Thus

$$\bar{S} \subseteq \{x \in X : dist(x,S) = 0\}.$$

Conversely, consider any $x \in X$ such that $dist(x,S) = 0$. If $x \in S$, then $x \in \bar{S}$ and we are done. Let $x \in S^c$, then $dist(y,S^c) = 0$ and hence $y \in \partial S \subseteq \bar{S}$. It follows that $\{x \in X : dist(x,S) = 0 \subseteq S$. Consequently,

$$\bar{S} \subseteq \{x \in X : dist(x,S) = 0\}.$$

Alternatively,

Consider

$$\bar{S} = S \cup \partial S = S \cup (\partial S \cap S^c)$$

$$= S \cup \{x \in S^c : dist(x,S) = 0 = dist(x,S^c)\}$$

$$= S \cup \{x \in S^c : dist(x,S) = 0\} [Since\ x \in S^c \Rightarrow dist(x,S^c) = 0]$$

$$= \{x \in S : dist(x,S) = 0\} \cup \{x \in S^c : dist(x,S) = 0\}$$

$$= \{x \in X : dist(x,S) = 0\}.$$

(ii) Consider

$$Ext(S) = (\bar{S})^c$$

$$= \{x \in X : dist(x,S) = 0\}^c [From\ (i)]$$

$$= \{x \in X : dist(x,S) > 0\}.$$

(iii) Consider

$$S^0 = ((S^c)^c)^0$$

$$= Ext(S^c) [\ From\ Theorem\ 4.2.8\ (ii)]$$

$$= \{x \in X : dist(x,S^c) > 0\}$$

Alternatively

Consider

$$S^0 = S \setminus \partial S = S \setminus (\partial S \cap S)$$

$$= S \setminus \{x \in S : dist(x,S) = 0 = dist(x,S^c)\}$$

$$= S \setminus \{x \in S : dist(x,S^c) = 0\} [Since\ x \in S \Rightarrow dist(x,S) = 0]$$

$$= \{x \in S : dist(x,S^c) > 0\}$$

$$= \{x \in X : dist(x,S^c) > 0\} [Since\ x \in X \setminus S \Rightarrow dist(x,S^c) = 0]$$ ■

Theorem 4.2.10 Let (X,d) be a metric space, $w \in X$ and S be a subset of X. Then (i). $diam(\bar{S}) = diam(S)$;

(ii). $dist(w,S) \leq dist(w,\partial S)$; and

(iii). $dist(w,\bar{S}) = dist(w,S)$.

Proof:(i) Since $S \subseteq \bar{S}$, therefore $diam(S) \leq diam(\bar{S})$. Consider any $\epsilon > 0$and $x,y \in \bar{S}$, then

$$dist(x,S) = 0 = dist(y,S)[\because \bar{S} = \{x \in X : dist(x,S) = 0\}].$$

As a consequence there exist $a,b \in S$ such that $d(x,a) < \epsilon/2$and $d(y,b) < \epsilon/2$. Now

$$\begin{aligned} d(x,y) &\leq d(x,a) + d(a,y) \\ &\leq d(x,a) + d(a,b) + d(b,y) \\ &< d(a,b) + \epsilon/2 + \epsilon/2 \\ &\leq diam(S) + \epsilon. \end{aligned}$$

Thus

$$\begin{aligned} & d(x,y) < diam\ (S) + \epsilon \quad \forall\ x,y \in \bar{S} \\ \Rightarrow\ & \sup_{x,y \in \bar{S}} d(x,y) < diam\ (S) + \epsilon \\ \Rightarrow\ & diam\ (\bar{S}) < diam\ (S) + \epsilon\ . \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get

$$diam(\bar{S}) \leq diam(S).$$

Thus it follows that $diam(\bar{S}) = diam(S)$.

(ii) If $\partial S = \emptyset$, then $dist(w,\partial S) = \infty$, whence$dist(w,S) \leq dist(w,\partial S)$ and we are through with this case. Let $\partial S \neq \emptyset$ and $z \in \partial S$ be any element.Then $dist(z,S) = 0$. Now

$$\begin{aligned} dist(w,S) &\leq d(w,z) + dist(z,S) [Theorem\ \ 4.3.2] \\ &= d(w,z). \qquad\qquad [\because\ dist(z,S) = 0\,] \end{aligned}$$

Since z is an arbitrary element in ∂S, therefore

$$dist(w,S) \leq \inf_{z \in \partial S} d(w,z) = dist(w,\partial S).$$

Hence (ii) follows.

(iii) Since $S \subseteq \bar{S}$, then from Theorem 3.5 it follows that

$$dist(w,\bar{S}) \leq dist(w,S).$$

Consider any $z \in \bar{S}$, from Theorem 4.2.9 $dist(z,S) = 0$. Therefore

$$dist(w,S) \leq d(w,z) + dist(z,S) = d(w,z).$$

Since z is an arbitrary element in $\bar{S}$, therefore

$$dist(w,S) \leq \inf_{z \in \bar{S}} d(w,z) = dist(w,\bar{S}). \qquad \blacksquare$$

Remark: It is important to note that if $S = \emptyset$or $w \in \partial S$, then obviously $dist(w,S) = dist(w,\partial S)$. Also, if we consider $X = \mathbb{R}$, $w = 1$ and $S = [0,2]$, clearly we have strict inequality in that case i.e., $dist(w,S) < dist(w,\partial S)$. What happens if $\partial S \neq \emptyset$and $w \in S^c \backslash \partial S$? Do we attain equality in this case? There is a temptation to respond with yes as answer. But as noticed earlier, sometimes our intuition might fail to be correct about boundary points.

Here also it's the same. We have an example of a metric space (X,d), a point w in X and a subset S of X with $\partial S \neq \emptyset$ and $w \in S^c \backslash \partial S$ such that $dist(w,S) < dist(w,\partial S)$.

Example 4.2.11 Consider the metric space (X,d) where $X = \{0\} \cup [1,2] \cup [4,5]$ and d is the metric on X induced by the usual metric of $\mathbb{R}$ i.e.,

$$d(x,y) = |x-y| \ \ \forall\, x,y \in X.$$

Let $w = 0$and $S = [1,2] \cup (4,5]$. Then clearly, $\partial S = \{4\}$. Also,$dist(w,S) = 1$and $dist(w,\partial S) = 4$.Hence $dist(w,S) < dist(w,\partial S)$. ■

Can we have a similar relations with interiors in place of closures as provided in Theorem 4.2.10? We will see that interiors do not behave the same way. Given a metric space X and a subset $A$of X, there is no guarantee that $diam(A^0) = diam(A)$ or for any x in X, $dist(x,A^0) = dist(x,A)$.

Example 4.2.12 Consider the real line $\mathbb{R}$ with usual metric$u$. We know that $\mathbb{Q}^0 = \emptyset$ and therefore $diam(\mathbb{Q}^0) = -\infty$. Also, $diam(\mathbb{Q}) = \infty$. Hence $diam(\mathbb{Q}^0) < diam(\mathbb{Q})$. Moreover for any $x$in $\mathbb{R}$, $dist(x,\mathbb{Q}^0) = \infty$ and $dist(x,\mathbb{Q}) = 0$. Thus $ist(x,\mathbb{Q}) < dist\,(x,\mathbb{Q}^0)$. ■

Theorem 4.2.13 Let (X,d) be a metric space and S be a subset of X. Then

(i). $S^0 \cap \partial S = \emptyset$;

(ii). $S^0 \cap Ext(S) = \emptyset$;

(iii). $Ext(S) \cap \partial S = \emptyset$;

(iv). $S^0 \cup \partial S \cup Ext(S) = X$

In other words, the interior, the boundary and the exterior of S are mutually disjoint and their union is equal to X.

Proof: Since $S^0 = S\backslash \partial S$, therefore $S^0 \cap \partial S = \emptyset$. Also, since $\bar{S} = S \cup \partial S$, therefore S^0 and ∂S are contained in $\bar{S}$.Hence $S^0 \cap Ext(S) = \emptyset$and $Ext(S) \cap \partial S = \emptyset$. Moreover

$$\begin{aligned} X &= \bar{S} \cup \bar{S}^c \\ &= (S \cup \partial S) \cup Ext(S) \\ &= \big((S\backslash \partial S) \cup \partial S\big) \cup Ext(S) \\ &= S^0 \cup \partial S \cup Ext(S). \end{aligned}$$

Hence the theorem follows. ■

4.3 INCLUSION OF CLOSURES AND INTERIORS

As seen, on the real line$\mathbb{R}$, boundary of $\mathbb{Q}$ is $\mathbb{R}$ and boundary of $\mathbb{R}$ is empty i.e, $\partial\mathbb{Q} = \mathbb{R}$ and $\partial\mathbb{R} = \phi$. Hence inclusion of one set into another does not imply the inclusion of boundary of one set into the boundary of the other seti.e, $A \subseteq B \nRightarrow \partial A \subseteq \partial B$.

In this subsection we try to look whether the inclusion of one set in another imply the inclusion of closure of one set into the closure of the other set? We shall examine a similar situation for interiors too.

Theorem 4.3.1 Let (X,d) be a metric space and $A \subseteq B \subseteq X$. Then

(i). $\bar{A} \subseteq \bar{B}$;

(ii). $A^0 \subseteq B^0$.

Proof: Consider

$$x \in \bar{A} \Leftrightarrow dist(x,A) = 0 \quad [Theorem\ 4.2.9]$$
$$\Rightarrow dist\ (x,B) = 0 \quad [\because A \subseteq B \Rightarrow dist\ (x,B) \le dist\ (x,A)]$$
$$\Leftrightarrow x \in \bar{B}\,.$$

Thus $\bar{A} \subseteq \bar{B}$.

Now

$$x \in A^0 \Leftrightarrow x \in A \ \ and\ x \notin \partial A$$
$$\Leftrightarrow x \in A \ \ and\ \ dist(x,A^c) > 0$$
$$\Rightarrow x \in B \ \ and\ \ dist\ (x,B^c) > 0 \quad [\because B^c \subseteq A^c \Rightarrow dist\ (x,A^c) \le dist\ (x,B^c)]$$
$$\Leftrightarrow x \in B \ \ and\ x \notin \partial B$$
$$\Leftrightarrow x \in B^0\ .$$

Thus $A^0 \subseteq B^0$. ■

Theorem 4.3.2 Let (X,d) be a metric space and S be a subset of X. Then

(i). $\partial(\bar{S}) \subseteq \bar{S}$;

(ii). $\partial(S^0) \cap S^0 = \emptyset$;

(iii). $\bar{\bar{S}} = \bar{S}$; and

(iv). $(S^0)^0 = S^0$.

Proof:(i) Let $x \in \partial(\bar{S})$ be any arbitrary element. Then

$$dist(x,\bar{S}) = 0 = dist(x,(\bar{S})^c)\,.$$

Since $dist(x,\bar{S}) = 0$, for any $\epsilon > 0$ there exists $y \in \bar{S}$ such that $d(x,y) < \epsilon$. Consider

$$dist(x,S) \ \le\ d(x,y) + dist(y,S)$$
$$= d(x,y) < \epsilon. \qquad [\because y \in \bar{S} \Rightarrow dist(y,S) = 0]$$

Since $\epsilon > 0$ is arbitrary, we get $dist(x,S) = 0$.

Now

$$S \subseteq \bar{S} \ \Rightarrow\ (\bar{S})^c \subseteq S^c$$
$$\Rightarrow\ dist(x,S^c) \le dist(x,(\bar{S})^c) = 0$$
$$\Rightarrow\ dist(x,S^c) = 0.$$

Thus $x \in \partial S \subseteq \bar{S}$. Since $x \in \partial(\bar{S})$ is arbitrary, $\partial(\bar{S}) \subseteq \bar{S}$.

(ii) Let $x \in S^0$, then $x \in S$ and $x \notin \partial S$. Consequently, $dist(x,S^c) > 0$. We claim that $dist(x,(S^0)^c) > 0$. Suppose on the contrary, $dist(x,(S^0)^c) = 0$. Then for any $\epsilon > 0$, there exists $y \in (S^0)^c$ such that

$$d(x,y) < dist(x,(S^0)^c) + \epsilon = \epsilon.$$

Now $y \in (S^0)^c = \overline{S^c}$ implies that $dist(y,S^c) = 0$. Thus

$$dist(x,S^c) \le d(x,y) + dist(y,S^c)\ = d(x,y) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get $dist(x, S^c) = 0$, a contradiction to the fact that $dist(x, S^c) > 0$. Hence our assumption is wrongand$dist(x, (S^0)^c) > 0$, as claimed. Consequently, $x \notin \partial(S^0)$. Since $x \in S^0$ is arbitrary, it follows that $\partial(S^0) \cap S^0 = \emptyset$.

(iii)$\bar{\bar{S}} = \bar{S} \cup \partial(\bar{S}) = \bar{S}[Since\ from\ (i)\ we\ have\ \partial(\bar{S}) \subseteq \bar{S}]$

(iv) $(S^0)^0 = S^0 \backslash \partial(S^0) = S^0[Since\ from\ (ii) we\ have\ \partial(S^0) \cap S^0 = \emptyset\ ]$. ∎

Important Note
In the proof of (i), we observed that $\partial(\bar{S}) \subseteq \partial S$ i.e., boundary of closure of a set is always contained in the boundary of the set. Also on the real line $(\mathbb{R}, u)$, $\partial(\bar{\mathbb{Q}}) = \emptyset \subset \partial\mathbb{Q} = \mathbb{R}$. Thus in general, boundary of closure of a set need not be equal to boundary of the set.

Corollary4.3.3 Let (X, d) be a metric space and S and A be subsets of X. Then

(i). If $S \subseteq A \subseteq \bar{S}$, then $\bar{A} = \bar{S}$.

(ii). If $S^0 \subseteq A \subseteq S$, then $A^0 = S^0$.

Proof: From Theorem 4.3.1 and Theorem 4.3.2 we havefollowing two implications

$$S \subseteq A \subseteq \bar{S} \implies \bar{S} \subseteq \bar{A} \subseteq \bar{\bar{S}} = \bar{S}$$
$$S^0 \subseteq A \subseteq S \implies (S^0)^0 \subseteq A^0 \subseteq S^0$$

Thus (i) and (ii) follows. ∎

Remark: If $S^0 \subseteq A \subseteq \bar{S}$, then in general, we may not have either $\bar{A} = \bar{S}$ or $A^0 = S^0$ or even both. For example consider the real line $\mathbb{R}$, since $\mathbb{Q}^0 = \emptyset$ and $\bar{\mathbb{Q}} = \mathbb{R}$, therefore

$$\mathbb{Q}^0 \subseteq (a, b) \subseteq \bar{\mathbb{Q}} \quad but\ clearly, \quad \mathbb{Q}^0 \neq (a, b) \quad and\ \bar{\mathbb{Q}} \neq \overline{(a, b)}.$$

5. SOLVED PROBLEMS

Problem 1. Let (X, d) be a metric space, A and B be subsets of X such that $\partial B \subseteq A \subseteq B$. Show that $\partial B \subseteq \partial A$.

Proof: Consider

$$x \in \partial B \Rightarrow x \in A \quad and \quad dist\ (x, B^c) = 0 \quad [\because\ \partial B \subseteq A]$$
$$\Rightarrow dist\ (x, A) = 0 \ \ and \ \ dist\ (x, A^c) = 0. \qquad [\because B^c \subseteq A^c \Rightarrow dist\ (x, A^c) \leq dist\ (x, B^c)]$$
$$\Rightarrow x \in \partial A\ .$$

It follows that $\partial B \subseteq \partial A$. ∎

Problem 2. Let (X, d) be a metric space, S be a subset of X and a be an isolated points of S. Show that a is a boundary point of S in X if and only if $a \notin iso(X)$.

Proof: Let $a \in \partial S$, then $dist(a, S) = 0 = dist(a, S^c)$. Now

$$a \in iso(S) \implies a \in S \implies a \notin S^c \implies dist(a, S^c \backslash \{a\}) = 0.$$

Thus $dist(a, X \backslash \{a\}) = 0$. Hence $a \notin iso(X)$.

Conversely, suppose $a \notin iso(X)$. Then $dist(a, X\backslash\{a\}) = 0$. Now since $a \in iso(S)$, therefore $dist(a, S\backslash\{a\}) > 0$. Then there exists $r \in \mathbb{R}^+$ such that $dist(a, S\backslash\{a\}) > r$. Since $dist(a, X\backslash\{a\}) = 0$or any ϵ $(0 < \epsilon < r)$, there exists $y \in X\backslash\{a\}$ such that $d(a, y) < \epsilon$. Clearly, $y \notin S$ and therefore $dist(a, S^c) \le d(a, y) < \epsilon$. Since ϵ is arbitrary, we get$dist(a, S^c) = 0$. Also, since $a \in S$ is an isolated point of S, therefore $dist(a, S) = 0$. Thus

$$dist(a, S) = 0 = dist(a, S^c) = 0, \qquad i.e., \quad a \in \partial S\,.$$

Alternative proof:

Now

$$\begin{aligned} a \in \partial S &\Leftrightarrow dist(a, S) = 0 = dist(a, S^c) \\ &\Leftrightarrow dist(a, S^c) = 0 \qquad\qquad [a \in iso(S) \Rightarrow a \in S \Rightarrow dist(a, S) = 0] \\ &\Leftrightarrow dist(a, S^c \setminus \{a\}) = 0 \qquad [\because S^c = S^c \setminus \{a\}]\,.--------- (\boldsymbol{A}) \end{aligned}$$

Claim: $dist(a, S^c \setminus \{a\}) = 0 \;\Leftrightarrow\; dist(a, X \setminus \{a\})$.

Since $S^c \setminus \{a\} \subseteq X \setminus \{a\}$, therefore $dist(a, S^c \setminus \{a\}) = 0 \Rightarrow dist(a, X \setminus \{a\}) = 0$. Conversely, suppose $dist(a, X \setminus \{a\}) = 0$. We will prove that$dist(a, S^c \setminus \{a\}) = 0$. Suppose on the contrary, $dist(a, S^c \setminus \{a\}) > 0$. Then there exists $k \in \mathbb{R}^+$ such that $dist(a, S^c \setminus \{a\}) > k$. Also, since $a \in iso(S)$, therefore $dist(a, S\backslash\{a\}) > 0$ and hence $r \in \mathbb{R}^+$ such that $dist(a, S\backslash\{a\}) > r$. Let $t = \min\{k, r\}$. Then

$$d(a, x) > t \quad \forall\, x \in X \setminus \{a\} [\because\; dist(a, S^c \setminus \{a\}) > t \quad and \quad dist(a, S \setminus \{a\}) > t]$$

$$\Rightarrow \quad dist(a, X \setminus \{a\}) = \inf_{x \in X \setminus \{a\}} d(a, x) \ge t > 0\,, \qquad contradiction.$$

Hence it follows that

$$dist(a, S^c \setminus \{a\}) = 0.$$

Thus

$$dist(a, S^c \setminus \{a\}) = 0 \;\Leftrightarrow\; dist(a, X \setminus \{a\}) = 0 \Leftrightarrow a \notin iso(X).------ (\boldsymbol{B})$$

Consequently, from ($\boldsymbol{A}$)and ($\boldsymbol{B}$), it follows that

$$a \in \partial S \Leftrightarrow a \notin iso(X). \qquad \blacksquare$$

Problem 3. Let (X, d) be a metric space and S be a subset of X. Show that

$$\bar{S} = acc(S) \cup iso(S).$$

Proof: Consider

$$\begin{aligned} & x \in iso(S) \\ \Leftrightarrow\;& x \in S \;\; and \;\; dist(x, S\backslash\{x\}) > 0 \\ \Leftrightarrow\;& x \in S \;\; and \;\; x \notin acc(S) \\ \Leftrightarrow\;& x \in S\backslash acc(S) \end{aligned}$$

Hence it follows that $iso(S) = S\backslash acc(S)$.Now from Theorem 4.2.7

$$\bar{S} = S \cup acc(S)$$

$$\Rightarrow \qquad \bar{S} = \big(S\backslash acc(S)\big) \cup acc(S)$$

$$\Rightarrow \qquad \bar{S} = iso(S) \cup acc(S). \qquad \blacksquare$$

Problem 4. Show that every countable subset of $\mathbb{R}$ has empty interior in $\mathbb{R}$ and is therefore included in its own boundary in $\mathbb{R}$.

Proof: Let $A = \{x_1, x_2, \dots\dots\}$ be a countable subset of $\mathbb{R}$. Without loss of generality we may assume that $\langle x_n \rangle$ is a monotonic sequence. Now consider any $k \in \mathbb{N}$ and let $r_k = |x_{k+1} - xk > 0$. Then for any ϵ $0<\epsilon<rk$,

$$y = x_k + \frac{(x_{k+1} - x_k)}{|x_{k+1} - x_k|} \epsilon$$

is in A^c. Thus it follows that $dist(x_k, A^c) \leq dist(x_k, y) \leq \epsilon$. Since ϵ is arbitrary, we get

$$dist(x_k, A^c) = 0.$$

Also, since $dist(x_k, A) = 0$ being in A, therefore $x_k \in \partial A$. Since $k \in \mathbb{N}$ was arbitrary, therefore $A \subseteq \partial A$. Hence $A^0 = A \backslash \partial A = \emptyset$.

■

Problem 5. Find a metric space in which no non-empty countable subset has empty interior.

Proof: Let (X, d) be any discrete metric space and S be any non-empty subset of X. We claim that $S^0 = S$. It is enough to prove that no point in S is a boundary point of S. Let $x \in S$ be any point. If $S^c = \phi$, then $dist\ (x, S^c) = \infty$ and hence $x \notin \partial S$. If $S^c \neq \phi$, then

$$d(x, y) = 1 \quad \forall \ y \in S^c$$
$$\Rightarrow \quad dist\ (x, S^c) = \inf_{y \in S^c} d(x, y) = 1$$
$$\Rightarrow \quad x \notin \partial S\,.$$

This it follows that every non-empty subset of X has non-empty interior. In particular no non-empty countable subset has empty interior. ■

Problem 6. Let (X, d) be a metric space and S be a subset of X. Show that

$$A^0 \cup \partial A = \bar{A} \text{ and } \partial A = \bar{A} \cap \overline{A^c}.$$

Proof: Consider

$$\bar{A} = A \cup \partial A$$
$$= (A \backslash \partial A) \cup \partial A$$
$$= A^0 \cup \partial A.$$

Also, since $\partial A = \partial(A^c)$ therefore

$$\overline{A^c} = A^c \cup \partial(A^c) = A^c \cup \partial A.$$

Now consider

$$\bar{A} \cap \overline{A^c} = (A \cup \partial A) \cap (A^c \cup \partial A)$$
$$= (A \cap A^c) \cup \partial A$$
$$= \emptyset \cup \partial A = \partial A.$$

■

Problem 7. Suppose X is a metric space and A is a subset of X. Is it necessarily the case that ∂A and $\partial(\bar{A})$ are identical?

Proof: No. For example on the real line $(\mathbb{R}, u)$,

$$\partial \mathbb{Q} = \mathbb{R} \ \ and \ \ \partial(\overline{\mathbb{Q}}) = \partial \mathbb{R} = \phi\ .$$ ■

Problem 8. Suppose X is a metric space and S is a subset of X. Show that $diam(S^0)$ need not be same as $diam(S)$.

Proof: On the real line $(\mathbb{R}, u)$,

$$diam\ (\mathbb{Q}^0) = diam\ (\phi) = -\infty \quad and \quad diam(\mathbb{Q}) = \infty\ . \qquad \blacksquare$$

6. SUMMARY

In this chapter we introduced the concept of boundary point of a set i.e., point in the space has distance zero both from the set and from its complement. Many interested examples related to boundary were discussed. We noticed that on the real line though boundary of the set of rationals $\mathbb{Q}$ and that of the irrationals $\mathbb{R} \setminus \mathbb{Q}$ is the complete set $\mathbb{R}$, but the set $\mathbb{R}$ itself has empty boundary. We then introduced the concepts of interior, exterior and closure of a set defined using the boundary of that set. Various interesting properties and example associated with these important concepts were discussed. Also, we tried to look how these concepts are related to themselves and how they are related to concepts like isolated points, accumulation points etc. studied earlier.

7. EXERCISES

Question 1. Verify that the graph $\Gamma = \{(x, \sin(1/x)) : x \in \mathbb{R}^+\}$ of $x \to \sin(1/x)$ defined on $\mathbb{R}\backslash\{0\}$ has boundary $\Gamma \cup \{(0, y) : y \in [-1, 1]\}$in $\mathbb{R}^2$.

Question 2. Suppose$a, b \in \mathbb{R}$and $a < b$. Show that the boundary points of theinterval(a, b) are$a$and b.

Question 3. Consider the metric space $(\mathcal{F}, d)$ where $\mathcal{F}$ is the set of all functions from $[0,1]$ to $[0,1]$ and d is the metric given by

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}.$$

Let $\mathcal{C}$denotethe collection of constant functions in$\mathcal{F}$. Show that$\partial\mathcal{C} = \mathcal{C}$.

Question 4. Suppose$n \in \mathbb{N}$ and, for each$i \in \mathbb{N}_n$, (X_i, τ_i) is a metric space. Let $d$be a conserving metric on$P = \prod_{i=1}^{n} X_i$. Suppose$S \subseteq P$. Explore the relationship between$\partial_P(S)$ and $\partial_{X_i}(\pi_i(S))$for $i \in \mathbb{N}_n$.

Question 5. Let (X, d)be a metric space and$A \subseteq B \subseteq X$. Prove that

$$dist\ (\bar{A}, \bar{B}) = dist\ (A, B)\ .$$

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