METRIC SPACES

TOPIC: DISTANCE

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" *Mathematics is the tool specially suited for dealing with abstracts concepts and there is no limit to its power.* "

Paul Dirac, 1902 – 1984

1.LEARNING OUTCOMES

At the end of this chapter, the reader will get familiarity with the concept of distances on a given abstract space and how we extend these notions from reals to any abstract space.We will do extensive study on notions like isolated points and accumulation points that originate from the concept of distance.

2. PREREQUISITES

We expect that the reader knows the definition of metric spaces and all the standard examples of metric spaces. To make the text self-contained we state the definition of a metric space.

Definition 2.1 Let X be a non-empty set and let $d : X \times X \to \mathbb{R}$ be the function satisfying the following conditions :

[P1] $d(x,y) \geq 0 \quad \forall\, x,y \in X$**[Non-negativity]**

[P2] $d(x,y) = 0 \iff x = y$;

[P3] $d(x,y) = d(y,x)$, for all x,y in X ;**[Symmetry Property]**

[P4] If $x,y,z$are any three elements of X, then

$$d(x,z) \leq d(x,y) + d(y,z). \qquad \text{[\textbf{Triangle Inequality}]}$$

Then d is called a **metric** on X and (X,d) is called a**metric space**. For each x,yin X, we call the number $d(x,y)$ the **distance** between x and y w.r.t the metric d.

Extended Real Numbers

The set $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is called the set of extended real numbers. With the convention that $-\infty < r < \infty$ for all $r \in \mathbb{R}$, we extend the ordering of $\mathbb{R}$ to $\widetilde{\mathbb{R}}$. Also, we extend addition and multiplication from $\mathbb{R}$ to $\widetilde{\mathbb{R}}$ as follows :

[A1] $r + \infty = \infty + r = \infty \quad for\ all r \in \mathbb{R} \cup \{\infty\}$

[A2] $r + (-\infty) = (-\infty) + r = -\infty \quad for\ all r \in \mathbb{R} \cup \{-\infty\}$

[M1] $r.\infty = \infty.r = \infty \quad for\ all r \in \mathbb{R}^+ \cup \{\infty\}$

[M2] $r.(-\infty) = (-\infty).r = -\infty \quad for\ all r \in \mathbb{R}^+ \cup \{\infty\}$

[M3] $r.\infty = \infty.r = -\infty \quad for\ all r \in \mathbb{R}^- \cup \{-\infty\}$

[M4] $r.(-\infty) = (-\infty).r = \infty \quad for\ all r \in \mathbb{R}^- \cup \{-\infty\}$

We also define $|\infty| = |-\infty| = \infty$, but we do not define following:

[U1] $\infty + (-\infty), \quad (-\infty) + \infty$**[Undefined]**

[U2] $0.\infty, \quad \infty.0, \quad (-\infty).0, \quad 0.(-\infty)$**[Undefined]**

Supremum and Infimum

In this chapter we will be using the notion of supremum and infimum of subset of reals repeatedly, thus we list some of the results to refresh the concept of reader.

Definition 2.2 Let S be any non-empty subset of $\mathbb{R}$.

1. A number $u \in \mathbb{R}$ is said to be an **upper bound of S** if $s \leq u$ for all $s \in S$.

2. The set S is **bounded above** if there exists an upper bound of$S$.
3. If the set S is bounded above, then a number $M \in \mathbb{R}$ is said to be **supremum (**or **least upper bound of S**), **denoted by $sup\, S$**if M is an upper bound of S and

$$M \leq u \quad for\ any\ upper\ bound\ \ u\ of\ S\ .$$

4. If the set S is not bounded above i.e, for each $K \in \mathbb{R}$ there exists $w \in S$ such that

$$K < w\ .$$

In that case we write $\mathbf{sup}\ \boldsymbol{S} = \infty$**.**

Note: By convention we take supremum of an empty set as $-\infty$ i.e., $\mathbf{sup}\ \boldsymbol{\phi} = -\infty$.

Theorem 2.3 An upper bound $M \in \mathbb{R}$ of a non-empty set $S \subseteq \mathbb{R}$ is the supremum of S if and only if for each $\epsilon > 0$, there exists $x_\epsilon \in S$ such that

$$M < x_\epsilon + \epsilon\ .$$

Proof: Let $M = \sup\ S$. Then for any real number $\epsilon > 0$, $M - \epsilon$ is not an upper bound of S. Therefore there exists $x_\epsilon \in S$ such that

$$M - \epsilon < x_\epsilon \quad i.e., \quad M < x_\epsilon + \epsilon\ .$$

Conversely,suppose hypothesis hold and$u$ be any upper bound of S. Consider any $\epsilon > 0$, then by hypothesis, there exists $x_\epsilon \in S$ such that

$$M < x_\epsilon + \epsilon \leq u + \epsilon\ .$$

Since $\epsilon > 0$ is arbitrary, $M \leq u$. Again since u is an arbitrary upper bound of S, therefore M is a least upper bound of S. Thus $M = \sup\ S$. ■

Definition 2.4 Let S be any non-empty subset of $\mathbb{R}$.

1. A number $u \in \mathbb{R}$ is said to be a**lower bound of S** if $u \leq s$ for all $s \in S$.
2. The set S is **bounded below** if there exists a lower bound of S.
3. If the set S is bounded below, then a number $v \in \mathbb{R}$ is said to be **infimum (**or**greatestlower bound of S**)if v is a lower bound of S and

$$v \geq w \quad for\ any\ lower\ bound\ \ w\ of\ S\ .$$

4. If the set S is not bounded below i.e., for each $K \in \mathbb{R}$, there exists $w \in S$ such that

$$w < K\ .$$

In that case we write $\mathbf{inf}\ \boldsymbol{S} = -\infty$

Note: By convention we take infimum of an empty set as ∞ i.e., $\mathbf{inf}\ \boldsymbol{\phi} = \infty$**.**

Theorem 2.5 A lower bound m of a non-empty set $S \subseteq \mathbb{R}$ is the infimum of S if and only if for each $\epsilon > 0$, there exists $x_\epsilon \in S$ such that

$$m + \epsilon > x_\epsilon\ .$$

Theorem 2.6 Let A and B be any non-empty subsets of $\mathbb{R}$ such that $A \subseteq B$. Then

1. $\sup\ A \leq \sup\ B$
2. $\inf\ B \leq \inf\ A$.

Proof: Since $a \in B$ for all $a \in A$, therefore $a \leq \sup B \ \ \forall a \in A$. Thus $\sup B$ is an upper bound of A, hence $\sup A \leq \sup B$.

Similarly, as $\inf B \leq a \ \forall a \in A$, $\inf B$ is a lower bound of A. Thus $\inf B \leq \inf A$. ■

Supremum and Infimum Relative to a subset of $\mathbb{R}$

Definition 2.7 Let $X \subseteq \mathbb{R}$ be any non-empty set and S be any non-empty subset of X.

1. A number $u \in X$ is said to be an **upper bound of S relative to X** if $\leq u \forall s \in S$.
2. The set S is **bounded above in X** if $\exists$ an upper bound of S relative to X.
3. If the set S is bounded above in X, then a number $M \in X$ is said to be **supremum (**or **least upper bound of S**) **relative to X, denoted by $sup_X\, S$** if M is an upper bound of S in X and
$$M \leq u \quad for\ any\ upper\ bound\ \ u\ of\ S\ relative\ to\ X.$$
4. If the set S is not bounded above i.e, for each $x \in X$ there exists $w \in S$ such that
$$x < w\,.$$
In that case we write $\mathbf{sup}\ \boldsymbol{S} = \infty$.

Similarly, we have the concept of infimum of a non-empty subset S of X (relative to X), denoted by $\mathbf{inf}_{\boldsymbol{X}}\ \boldsymbol{S}$. Further by convention we take

$$\boldsymbol{sup_X\, \phi = -\infty \quad and \quad inf_X\, \phi = \infty}\,.$$

3. DISTANCE

Wealready know that with the help of a metric we extend the concept of distance inreals to any given abstract set. In a way we can say that, metrics are designed to measure distance between two points of any given abstract set.

3.1 DIAMETER

In the Cartesian plane, the diameter of a circle $C$is the maximum distance between its points i.e., $diam(C) = \max\{d(u,v) : u, v \in C\}$, where $d(u,v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$is the usual distance between two points in Cartesian plane. Using the concept of metric we can easily extend this definition of diameter of a circle to define diameters for any arbitrary set in an arbitrary metric space. Since in the general case maximum may not exist, therefore we will settle for the supremum of the various distances between points of the set.

Definition 3.1.1 Let (X, d) be a metric space and A be any non-empty subset of X. Then the diameter of set A, denoted by $diam(A)$, is defined to be

$$diam(A) = \sup\{\, d(u,v) :\ u, v\ \in A\,\}.$$

Since diameter of a set by definition is metric dependent and we can have several metrics defined on a same set, thus to avoid any ambiguity we sometimes use$diam_d(A)$. Here $diam_d(A)$ means diameter of set A w.r.t metric d.

By convention the diameter of the empty set is defined to be $\sup \emptyset\, (= -\infty)$ i.e.,

$$\boldsymbol{diam\,(\phi) = -\infty}\,.$$

> **Let (X,d) be a metric space, A be any non-empty subset of X and $\epsilon > 0$ be any real number. Then, by Theorem 2.3, $\exists\, u, v \in A$ such that**
>
> $$\boldsymbol{diam\ (A) < d(u,v) + \epsilon \qquad provided\ \ diam\ (A) < \infty}$$

Remarks: Given any metric space (X, d),

1. Diameter of all singleton subsets of X is 0.$[\because\ d(u,u) = 0]$

2. Subset of X consisting of at least two elements has diameter in $\mathbb{R}^+ \cup \{\infty\}$.

Example 3.1.2 (Diameter of Unit Circle in $\mathbb{R}^2$)

Consider the real line $(\mathbb{R}^2, d)$ with d as the Euclidean metric on $\mathbb{R}^2$ and the unit circle

$$C = \{(x,y) \in \mathbb{R}^2\ :\ x^2 + y^2 = 1\}.$$

From our basic knowledge we know that diameter of the unit circle C is 2. Now let us verify this fact from our metric definition. Clearly, by definition of C, points $a = (1,0), b = (-1,0) \in C$. Thus

$$diam\ (C) \geq d(a,b) = \sqrt{\big(1-(-1)\big)^2 + (0-0)^2} = \sqrt{4} = 2\,.$$

To see that $diam\,(C)$ is indeed equal to 2, consider any $\epsilon > 0$. Then $\exists\ w_1 = (x_1, y_1)$, $w_2 = (x_2, y_2) \in C$ such that

$$\begin{aligned}
diam\ (C) &< d(w_1, w_2) + \epsilon \\
&= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\ + \epsilon \\
&= \sqrt{x_1^2 + y_1^2 - 2x_1y_1 + x_2^2 + y_2^2 - 2x_2y_2}\ + \epsilon \\
&= \sqrt{2 - (2x_1y_1 + 2x_2y_2)}\ + \epsilon \\
&\leq \sqrt{2 + x_1^2 + y_1^2 + x_2^2 + y_2^2}\ + \epsilon \qquad [\because (a+b)^2 \geq 0 \Rightarrow a^2 + b^2 \geq -2ab] \\
&= \sqrt{4} + \epsilon = 2 + \epsilon
\end{aligned}$$

Thus $diam\ (C) < 2 + \epsilon \quad \forall\, \epsilon > 0$. Hence it follows that $diam\ (C) \leq 2$. Consequently,

$$diam(C) = 2.$$

Again let us consider the unit circle $C = \{(x,y) \in \mathbb{R}^2\ :\ x^2 + y^2 = 1\}$in $\mathbb{R}^2$. But this time let us take discrete metricd^*on $\mathbb{R}^2$, instead of our usual metric. Is the diameter of the circle again 2? No, $diam_{d^*}(C) = 1$. In fact, w.r.t discrete metric,diameter of any set S with $|S| \geq 2$ is exactly 1. Thus we see that diameter of a set may be different w.r.t different metrics.

> **Let(X,d) discrete metric spacewith $|X| \geq 2$ and $S \subseteq X$ be any set. Then**
>
> $$\boldsymbol{diam\ (S) = \begin{cases} -\infty & if\ |S| = 0 \\ 0 & if\ |S| = 1 \\ 1 & if\ |S| \geq 2 \end{cases}.}$$

Theorem 3.1.3 Let(X, d)be a metric space and $A \subseteq B \subseteq X$.Then $diam(A) \leq diam(B)$.

Proof: Now

$$\begin{aligned}
A \subseteq B &\Rightarrow \{d(u,v) : u, v\ \in A\} \subseteq \{d(x,y) : x, y\ \in B\} \\
&\Rightarrow \sup\{d(u,v) : u,v\ \in A\} \leq \sup\{d(x,y) : x,y\ \in B\}\,[From\ Theorem\ 2.6]
\end{aligned}$$

$$\Rightarrow diam(A) \leq diam(B)\,. \qquad \blacksquare$$

Example3.1.4 Consider the metric space $(\mathcal{F}, d)$ where $\mathcal{F}$ is the set of all functions from $[0,1]$ to$[0,1]$ and d is the metric given by

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}.$$

Let S be any subset of $\mathcal{F}$. Then for any $f, g \in S$,

$$\begin{aligned}
& f(x), g(x) \in [0,1] && \forall\, x \in [0,1] \\
\Rightarrow\quad & -1 \leq f(x) - g(x) \leq 1 && \forall\, x \in [0,1] \\
\Rightarrow\quad & 0 \leq |f(x) - g(x)| \leq 1 && \forall\, x \in [0,1] \\
\Rightarrow\quad & 0 \leq \sup\{|f(x) - g(x)| : x \in [0,1]\} \leq 1 && \\
\Rightarrow\quad & 0 \leq d(f,g) \leq 1\,. && \forall\, f, g \in S \\
\Rightarrow\quad & 0 \leq \sup\{d(f,g) \;:\; f, g \in S\} \leq 1 && \\
\Rightarrow\quad & 0 \leq diam\,(S) \leq 1. &&
\end{aligned}$$

Therefore for any subset $S$of $\mathcal{F}$,

$$0 \leq diam\,(S) \leq 1. \quad -------------- (\boldsymbol{A})$$

Since the constant functions$\mathbf{1}, \mathbf{0} \in \mathcal{F}$, where

$$\left.\begin{matrix} \mathbf{1}(x) = 1 \\ \mathbf{0}(x) = 0 \end{matrix}\right\} \quad \forall\, x \in [0,1]$$

we have

$$\begin{aligned}
diam\,(\mathcal{F}) &\geq d(\mathbf{1}, \mathbf{0}) \\
&= \sup\{|\mathbf{1}(x) - \mathbf{0}(x)| \;:\; x \in [0,1]\} \\
&= 1.
\end{aligned}$$

It follows that $diam\,(\mathcal{F}) \geq 1$. Hence,from $(\boldsymbol{A})$, we get $diam\,(\mathcal{F}) = 1$.

Now consider the set $\mathcal{C}$of all constant functions from $[0,1]$to$[0,1]$. Then obviously, $\mathcal{C} \subseteq \mathcal{F}$ and hence by Theorem 3.1.3

$$diam\,(\mathcal{C}) \leq diam\,(\mathcal{F}) = 1.$$

Also, since the constant functions $\mathbf{1}, \mathbf{0} \in \mathcal{C}$, therefore $diam\,(\mathcal{C}) \geq 1$. Consequently,

$$diam\,(\mathcal{C}) = 1\;. \qquad \blacksquare$$

NOTE

With the convention that $-\infty < x < \infty$, for any x in $\mathbb{R}^+$. The set$\mathbb{R}^+ \cup \{-\infty, 0, \infty\}$ under usual $\leq$ forms a poset (partially ordered set). Also, for any set X, the power set $\mathcal{P}(X)$ of X under $\subseteq$ forms a poset.

Now given any metric space (X, d), define a function

$$\boldsymbol{\Phi} : \mathcal{P}(X) \longrightarrow \mathbb{R}^+ \cup \{-\infty, 0, \infty\} \quad as$$

$$\boldsymbol{\Phi}(A) = diam\,(A).$$

Then by theorem 3.1.3 we notice that the function Φ is an isotone

(order preserving) from poset $(\mathcal{P}(X), \subseteq)$ into the poset $(\mathbb{R}^+ \cup \{-\infty, 0, \infty\}, \leq)$.

Theorem 3.1.5 For any subset S of $\mathbb{R}$, $diam(S) = \sup S - \inf S$.

Proof: **Case I** $S = \emptyset$

Then $diam(S) = -\infty, \quad \sup S = -\infty \quad and \quad \inf S = \infty$. Thus, in this case, $diam(S) = \sup S - \inf S$.

Case II $S \neq \emptyset$.

Sub case I $\sup S = \infty$

Then for any $b \in S$ and any given $\in \mathbb{R}^+$, $b + K \in \mathbb{R}$ and therefore there exists $a \in S$ such that $a > b + K$ (***see Definition* 2.2**). Thus

$$diam(S) = \sup\{|u - v| \ : \ u, v \in S\} \\ \geq |a - b| = a - b > K .$$

Since $K \in \mathbb{R}^+$ is arbitrary, therefore $diam(S) = \infty$. It follows that

$$diam\,(S) = \sup S - \inf S .$$

Sub case II $\inf S = -\infty$

Again for any $b \in S$ and any $K \in \mathbb{R}^+$, $b - K \in \mathbb{R}$ and therefore there exists $a \in S$ such that $a < b - K$. Thus

$$diam(S) = \sup\{|u - v| \ : \ u, v \in S\} \\ \geq |a - b| = b - a > K .$$

Since $K \in \mathbb{R}^+$ is arbitrary, therefore $diam(S) = \infty$. Hence

$$diam\,(S) = \sup S - \inf S .$$

Sub case III $-\infty < \inf S \leq \sup S < \infty$

Let $\epsilon > 0$ be any real number. By definition of infimum and supremum $\exists\, a, b \in S$ such that $\sup S - \epsilon/2 < a$ and $\inf S + \epsilon/2 > b$. Thus

$$diam(S) = \sup\{|u - v| \ : \ u, v \in S\} \\ \geq |a - b| \geq a - b$$

$$> \sup S - \epsilon/2 - (\inf\, S + \epsilon/2)$$

$$= \sup S - \inf S - \epsilon.$$

Since $\epsilon \in \mathbb{R}^+$ is arbitrary, we get

$$diam(S) \geq \sup S - \inf S .$$

Also, for any $x, y \in S$ we have $\inf\, S \leq x, y \leq \sup S$. Therefore

$$x - y \ \leq \ \sup S - \inf S \quad and \quad y - x \leq \ \sup S - \inf S$$

$$which \ \ implies \ \ that \ |x - y| \leq \ \sup S - \inf S.$$

Since $x, y \in S$ are arbitrary, therefore

$$\sup_{x,y\,\in S} |x - y| \leq \ \sup S - \inf S$$

$$i.e., \qquad diam(S) \leq \ \sup S - \inf S.$$

Hence $diam(S) = \ \sup S - \inf S$.

Thus in all the cases we get the desired equality and the theorem follows. ■

Corollary 3.1.6 For any$a, b \in \mathbb{R}$such that$a < b$, all the intervals $(a, b), [a, b), (a, b]$ and $[a, b]$ have diameter $b - a$.

Corollary 3.1.7 For any $a \in \mathbb{R}$,

$$diam\ (I) = \infty \quad for\ any\ interval\ \ I \in \{(a, \infty),\ [a, \infty),\ (-\infty, a),\ (-\infty, a],\ (-\infty, \infty)\}$$

$$diam\ ([a, a]) = 0\ .$$ ■

3.2 DISTANCES FROM POINTS TO SETS

Intuitively, we should define the distance from a point x to a non-empty set A in a metric space as the distance from x to the point in$A$ nearest to x. But, unfortunately, there may be cases in which nearest point of A to x may not exist. Consider the real line $\mathbb{R}$ with usual metric and take$A = (1,2)$and $x = 0$. Then it is easy to see that given any point $a \in A$, there exists $b \in A$ such that $d(a, x) < d(b, x)$ and hence nearest point of A to $x$does not exist in this case. Also, intuitively we can observe that distance from 0 to $(1,2)$ should be 1. We can see that 1 is the infimum of the distances from 0 to various points of $(1,2)$. We borrow this idea from the real line to more general setting.

Definition 3.2.1 Let (X, d) be a metric space, A be a subset of X and x be any point in X. The distance from $x$to A, denoted by $dist(x, A)$, is defined as

$$dist(x, A) = \inf\ \{d(x, a) :\ x\ \in\ A\ \}.$$

Since this is dependent on the metric being used, in order to avoid ambiguity, wesometimes denote it as $dist_d(x, A)$ to specify the metric under consideration.

Let (X, d) be a metric space, A be any non-empty subset of X and x be any point in X. Then for any real number $\epsilon > 0$, by Theorem 2.5, $\exists\ u \in A$ such that

$$\boldsymbol{d(u, x) < dist\ (x, A) + \epsilon}\ .$$

If $A = \phi$, then

$$\{d(a, x)\ :\ a \in A\ \} = \phi$$
$$\Rightarrow\quad \inf\{d(a, x)\ :\ a \in A\ \} = \infty$$
$$\Rightarrow\quad dist\ (x, \phi) = \infty$$

If $A \neq \phi$, then

$$0 \le dist\ (x, A) < \infty$$

Further if $x \in A$, then

$$dist\ (x, A) = \inf\{d(a, x)\ : a \in A\}$$
$$\le d(x, x) = 0$$
$$\Rightarrow \qquad dist\ (x, A) = 0\ .$$

Note that if $x \in A$, then $dist\ (x, A) = 0$. But if $dist\ (x, A) = 0$ need not imply $x \in A$. In the following example this point is illustrated.

Example 3.2.2 For any x in $\mathbb{R}$,

$$\text{dist}\ (\text{x}, \mathbb{Q}) = 0 \quad \text{and} \qquad \text{dist}(\text{x}, \mathbb{R}/\mathbb{Q}) = 0\ .$$

Consider any positive real number ϵ. Since every interval contains infinitely many rational numbers, therefore $\mathbb{Q} \cap (x, x+\epsilon) \neq \emptyset$. Choose any $y \in \mathbb{Q} \cap (x, x+\epsilon)$, then

$$dist\,(x,\mathbb{Q}) = \inf\{d(x,a) \;:\; a \in \mathbb{Q}\} \;\leq\; d(x,y) \;= |x-y| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get $dist(x,\mathbb{Q}) = 0$. Similarly we can show that for any x in $\mathbb{R}$, $dist(x,\mathbb{R}/\mathbb{Q}) = 0$. This example also illustrates the fact that distance of a point from a given set may be zero even when the point is outside that set. ■

$$\boldsymbol{dist(x,\mathbb{Q}) = 0 \text{ and } dist(x,\mathbb{R}/\mathbb{Q}) = 0 \quad \forall\; x \in \mathbb{R}}$$

Given any metric space (X,d) and any subset S of X such that $\sup S \in X$, then can we conclude something about $dist\,(\sup S\,,S)$? Is it necessarily zero ?

Since supremum of any set is defined in terms of the order, not on the metric considered, therefore there is no immediate reason to jump to any conclusion. In fact there may be occasions where supremum of a set may not exist and even if it exists then we may have $dist\,(\sup S\,,S) = 0$ or $dist\,(\sup S\,,S) \neq 0$. This fact is illustrated in the following example.

Example 3.2.3 Consider the metric space (X,d), where $X = [0,1) \cup \{2\}$ and $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ is the metric on X defined as

$$d(x,y) = |x-y|.$$

Then for the set $S = [1/2\,,1)$,

$$\sup_X S = 2 \qquad (\text{w.r.t poset } (X,\leq)).$$

Clearly, in this case

$$dist\,(\sup_X S\,,S) = dist\,(2,S) = \inf\{\, d(2,y) \;:\; y \in S\,\} = 1 \neq 0.$$

Now we consider the metric space (X^*,d^*), where $X^* = [0,1]$ and d^* is the usual distance metric on X^* i.e.,

$$d^*(x,y) = |x-y| \quad \forall\; x,y \in X^*.$$

Then for the set $S = [1/2\,,1)$,

$$\sup_{X^*} S = 1 \quad (\text{w.r.t poset } (X^*,\leq)).$$

Clearly, $dist\,(\sup_{X^*} S\,,S) = 0.$ ■

Now we consider the metric space $(\mathbb{R},u)$ with its usual metric. Since for any $a,b \in \mathbb{R}$

$$a \geq b \quad \Leftrightarrow \quad |a-b| = a-b \quad \Leftrightarrow \quad u(a,b) = a-b.$$

Thus we see that the usual metric on $\mathbb{R}$ is strongly related with the usual ordering in $\mathbb{R}$. This knowledge of $\mathbb{R}$ helps us in proving an important and beautiful consequence "*distance from the supremum of a subset of* $\mathbb{R}$ *to the subset itself is zero, provided supremum exists finitely*".

Theorem 3.2.4 Suppose S is a subset of $\mathbb{R}$ and $z \in \mathbb{R}$.

(i). $dist\,(z,S) \leq |z - \sup S\,|$ with equality if $z \geq \sup S$.

(ii). $dist\,(z,S) \leq |z - \inf S\,|$ with equality if $z \leq \inf S$.

(iii). If $\sup S \in \mathbb{R}$, then $dist\,(\sup S\,,S) = 0$.

(iv). If $\inf S \in \mathbb{R}$, then $dist\,(\inf S\,,S) = 0$.

Proof: We first prove part (i) of theorem by considering three cases separately.

Case 1. $S = \emptyset$

Then $dist\,(z,S) = \infty$. Also, by convention $\sup S = -\infty$. Thus

$$z > \sup\, S \quad and \quad |z - \sup S| = z - \sup\, S = z - (-\infty) = \infty.$$

Hence in this case $dist\,(z,S) = |z - \sup\, S|$ and $z > \sup\, S$.

Case 2. $S \neq \phi$ and $z \geq \sup\, S$

Let $\epsilon > 0$ be any real no. Then, by Theorem 2.3, there exists $v \in S$ such that $v \geq \sup S - \epsilon$. Consider

$$\begin{aligned} dist(z,S) &\leq d(z,v) \\ &= |z - v| \\ &= z - v \qquad\qquad [\because z \geq \sup\, S \geq v] \\ &\leq z - \sup S + \epsilon \\ &= |z - \sup S| + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get

$$dist(z,S) \leq |z - \sup S|.$$

Moreover, for any $s \in S$

$$d(z,s) = z - s \geq z - \sup S = |z - \sup S|.$$

Taking infimum over all s in S, we get

$$dist(z,S) = \inf\{d(z,s) : s \in S\} \geq |z - \sup S|.$$

Thus it follows that $dist(z,S) = |z - \sup S|$.

Case 3. $S \neq \emptyset$ and $z < \sup\, S$

Then by the definition of supremum, there exists $u \in S$ such that $z < u \leq \sup S$. Then

$$\begin{aligned} dist(z,S) &\leq d(z,u) \\ &= |z - u| \\ &= u - z \\ &\leq \sup S - z \\ &= |z - \sup S|. \end{aligned}$$

Hence $dist\,(z,S) \leq |z - \sup\, S|$.

From all the three cases (i) follows.

Similarly, we can show that (ii) holds. Furthersince $\sup\, S \in \mathbb{R}$ or $\inf\, S \in \mathbb{R}$ implies that $S \neq \phi$, therefore (iii) follows from (i) by simply putting $z = \sup S$, and (iv) follows by using $z = \inf S$ in (ii). Hence the theorem. ■

3.3 INEQUALITIES FOR DISTANCES

In this section, we discuss some useful inequalities related with distance between points and sets, which we derive using triangle inequality.

Theorem 3.3.1 Suppose X is a metric space, $x \in X$ and A and B are non-empty subsets of X for which $A \subseteq B$. Then

$$dist\,(x,B) \;\leq\; dist\,(x,A) \;\leq\; dist(x,B) + diam(B)\ .$$

Proof: Since $A \subseteq B$,

$$\{d(x,a) : a \in A\} \subseteq \{d(x,b) : b \in B\}.$$

It follows that

$$\inf\{d(x,a) : a \in A\} \geq \inf\{d(x,b) : b \in B\} \; [Theorem\ 2.6]$$
$$i.e., \quad dist\,(x,A) \geq dist(x,B).$$

Now for the next inequality consider any $a \in A$ and $b \in B$. Then

$$\begin{aligned} dist\,(x,A) &= \inf\{d(x,a) : a \in A\} \\ &\leq d(x,a) \\ &\leq d(x,b) + d(b,a) [using\ triangle\ inequality] \\ &\leq d(x,b) + diam(B) [\because b,a \in B\ and\ d(b,a) \leq diam(B)]. \end{aligned}$$

Therefore it follows that $dist(x,A) \leq d(x,b) + diam(B)$ for all $b \in B$. Thus taking infimum over all b in B, we get $dist(x,A) \leq \inf\{d(x,b) : b \in B\} + diam(B)$. Hence the desired inequality $dist(x,A) \leq dist(x,B) + diam(B)$. ■

Let (X,d) be any metric space and $A \subseteq B \subseteq X$. Then

$$\boldsymbol{dist\,(x,B) \leq dist\,(x,A) \leq dist(x,B) + diam(B) \quad \forall\, x \in X.}$$

Theorem 3.3.2 Suppose (X,d) is a metric space, $a,b \in X$ and S is a non-empty subset of X. Then

(i). $dist(a,S) \leq d(a,b) + dist(b,S)$; and

(ii). $|\,dist(a,S) - dist(b,S)| \leq d(a,b) \leq dist(a,S) + diam(S) + dist(b,S)$.

Proof: Consider any $x \in S$, then using triangle inequality we have

$$d(a,x) \leq d(a,b) + d(b,x).$$

Since $x \in S$ is arbitrary, taking infimum over all x in S we get

$$\inf\{d(a,x) : x \in S\} \leq d(a,b) + \inf\{d(b,x) : x \in S\}$$

$$i.e., \quad dist(a,S) \leq d(a,b) + dist(b,S). \qquad \ldots\ldots\ldots\ldots\ldots (\boldsymbol{A})$$

Hence (i) holds.

Now interchanging the role of a and b, we get

$$dist(b,S) \leq d(b,a) + dist(a,S). \qquad \ldots\ldots\ldots\ldots\ldots (\boldsymbol{B})$$

It follows that

$$dist\,(a,S) - dist(b,S) \leq d(a,b) [From\ (A)]$$
$$dist(b,S) - dist(a,S) \leq d(b,a) = d(a,b), \qquad [From\ (B)]$$

which together yield

$$|\,dist(a,S) - dist(b,S)| \leq d(a,b).$$

For the second inequality consider for any $x,y \in S$,

$$\begin{aligned} d(a,b) &\leq d(a,x) + d(x,b) \\ &\leq d(a,x) + d(x,y) + d(y,b) [\because d(x,b) \leq d(x,y) + d(y,b)] \\ &\leq d(a,x) + diam(S) + d(y,b) [\because d(x,y) \leq diam(S)] \end{aligned}$$

Taking infimum over all x in S, we get

$$d(a,b) \leq \inf\{d(a,x) : x \in S\} + diam(S) + d(y,b)$$

Taking infimum over all $y$in S, we get

$$d(a,b) \leq \inf\{d(a,x) : x \in S\} + diam(S) + \inf\{d(y,b) : y \in S\}$$

$$\Rightarrow \quad d(a,b) \leq dist(a,S) + diam(S) + dist(b,S).$$

Hence the theorem follows. ∎

3.4 ISOLATED POINTS

Intuitively, isolated means separate, not in touch with anyone. For example consider the subset $S = [1,2] \cup \{3\} \cup [4,5]$of $\mathbb{R}$, we see that 3 is not in touch with any member of the set $S$and**distance of 3 from all other points of $S$is at least 1**. In this case we can say that 3 is an isolated point of S. Extending this intuitive idea for isolated points, we define it for any general metric space.

Definition 3.4.1 Let (X,d) be a metric space, S be a subset of $X$and $z \in S$. Then z i an **isolated point** of S if distance of z from the rest of the points of S is non-zero i.e.,

$$dist(z, S\backslash\{z\}) \neq 0.$$

In this case, we say that z is isolated in S. The collection of all isolated points of S will be denoted by $iso(S)$.

Example 3.4.2 Consider the discrete metric space (X,d), where $d : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ is defined as

$$d(x,y) = \begin{cases} 1, & if\ x \neq y \\ 0, & if\ x = y. \end{cases}$$

Then every point in the space is at distance one from the rest of the points in the space. Hence it follows that every point in the space is an isolated point.

Every point in any set in a discrete metric space is an isolated point of that set.

Example 3.4.3 Consider the metric space $(\mathbb{R},u)$ with usual metric. We will show that every point in $\mathbb{N}$ of $\mathbb{R}$ is an isolated point of $\mathbb{N}$.
Let $n \in \mathbb{N}$ be any point. Then clearly, for any $k \in \mathbb{N}\backslash\{n\}$

$$u(n,k) = |n-k| \geq 1.$$

Taking infimum over all $k \in \mathbb{N}\backslash\{n\}$, we get

$$dist(n, \mathbb{N}\backslash\{n\}) \geq 1.$$

Also, since $u(n,n+1) = 1$, therefore it follows that $dist(n, \mathbb{N}\backslash\{n\}) = 1$. Hence **every point in the subset $\mathbb{N}$ of $\mathbb{R}$ is an isolated point.**

If we consider the metric space $(\mathbb{N},u)$ with u as the usual distance metric. Then for any subset $S$of $\mathbb{N}$, every point of S is an isolated point.

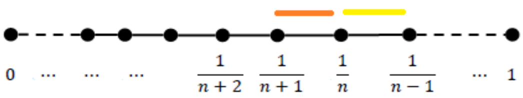
Example 3.4.4 Consider the metric space (X,d), where $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and $d: X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ is defined as

$$d(x,y) = |x-y| \qquad \forall\ x,y \in X$$

We will show that $iso(X) = \{1/n : \ n \in \mathbb{N}\}$.

Let $n \in \mathbb{N}$ be any natural number. If $n = 1$, then $1/2$ is the closest element of X to n. Also, if n>1, then obviously

$$0 < \cdots < \frac{1}{n+2} < \frac{1}{n+1} < \frac{1}{n} < \frac{1}{n-1} < \cdots < 1.$$



Thus it is easy to see that

$$dist\,(1/n, X\backslash\{1/n\}) = \begin{cases} \min\left\{\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\right\} & if\ n > 1 \\ \frac{1}{2} & if\ n = 1 \end{cases}$$

$$= \begin{cases} \min\left\{\frac{1}{n(n+1)}, \frac{1}{(n-1)n}\right\} & if\ n > 1 \\ \frac{1}{1(1+1)} & if\ n = 1 \end{cases}$$

$$= \frac{1}{n(n+1)}.$$

Thus for each $n \in \mathbb{N}$,

$$dist(1/n, X\backslash\{1/n\}) = \frac{1}{n(n+1)} > 0.$$

Hence it follows that$1/n$ is an isolated point of X for each $n \in \mathbb{N}$.

Next we have

$$dist(0, X\backslash\{0\}) = \inf\,\{d(0, 1/n) : \ n \ \in \ \mathbb{N}\}$$
$$= \inf\{1/n : \ n \ \in \ \mathbb{N}\}$$
$$= 0.$$

Therefore 0 is not an isolated point of X. Thus every point of X except 0 is an isolated point of X. ■

Question: Prove that $\inf\{1/n : \ n \ \in \ \mathbb{N}\} = 0$.

Example3.4.5 For any $x \in \mathbb{Q}$,

$$dist\,(x, \mathbb{Q} \setminus \{x\}) = 0$$

Consider any positive real number ϵ. Since every interval contains infinitely many rational numbers, therefore $\mathbb{Q} \cap (x, x+\epsilon) \neq \emptyset$. Choose any $y \in \mathbb{Q} \cap (x, x+\epsilon)$, then

$$dist\,(x, \mathbb{Q} \setminus \{x\}) = \inf\{d(x,a) \ : \ a \in \mathbb{Q} \setminus \{x\}\}$$
$$\leq d(x,y)$$
$$= |x - y|$$
$$< \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get $dist(x, \mathbb{Q} \setminus \{x\}) = 0$. Thus x is not an isolated point of $\mathbb{Q}$ for every point x in $\mathbb{Q}$. Hence$\mathbb{Q}$ has no isolated points i.e., $iso(\mathbb{Q}) = \phi$.

Similarly we can show that for any x in $\mathbb{I}$, $dist(x, \mathbb{I} \setminus \{x\}) = 0$, where $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$. ∎

Theorem 3.4.6 Suppose (X,d) is a metric space and F is a non-empty finite subset of X. Then every point of F is isolated in F.

Proof: If $F = \{a\}$, then $dist(a, F\setminus\{a\}) = \inf \emptyset = \infty$. In this case a (the only point of F) is an isolated point of F. Let $|F| \geq 2$ and $a \in F$ be any arbitrary point. Then $\{d(a,b) : b \in F\setminus\{a\}\}$ is a finite subset of $\mathbb{R}^+$ and hence attains its minimum i.e., $\exists\, c \in F\setminus\{a\}$ such that

$$\text{dist}\,(a, F\setminus\{a\}) = \min\{d(a,b) : b \in F\setminus\{a\}\} = d(a,c) \neq 0$$

Hence a is an isolated point of F. Since $a \in F$ is an arbitrary point, therefore every point of F is isolated in F. ∎

Theorem 3.4.7 Suppose (X,d) is a metric space and $A \subseteq B \subseteq X$. Then $A \cap iso(B) \subseteq iso(A)$.

Proof: Let

$$\begin{aligned} & z \in A \cap iso(B) \\ \Rightarrow\ & z \in A \ \ and\ \ z \in iso(B) \\ \Rightarrow\ & dist(z, B\setminus\{z\}) \neq 0\,. \end{aligned}$$

Since $A \subseteq B$, from Theorem 3.3.1 $dist(z, B\setminus\{z\}) \leq dist(z, A\setminus\{z\})$. Hence $dist(z, A\setminus\{z\}) \neq 0$ and $z \in iso(A)$. Thus $A \cap iso(B) \subseteq iso(A)$. ∎

3.5 ACCUMULATION POINTS

Roughly speaking, a point in a space is an accumulation point of a set if there are infinitely many points of the set surrounding the point in very close proximity. It can also be perceived as a point which has zero distance from the most of the points in the set. With this idea in mind, we now define accumulation point of a set properly.

Definition 3.5.1 Suppose X is a metric space, $z \in X$ and S is a subset of X. Then z is called an **accumulation point** or a **limit point** of S in X if

$$dist(z, S \setminus \{z\}) = 0\,.$$

The collection of all accumulation points of S in X will be denoted by $acc(S)$ or by $acc_X(S)$ if there is necessity of specifying the space under consideration.

In the last section we discussed isolated points. By definition, an isolated point of a set cannot be an accumulation point and vice versa. Further, consider any set S in a given metric space (X,d). Then for any point $x \in S$, we have two possibilities,

$$\begin{array}{cccc} & dist\,(x, S \setminus \{x\}) = 0 & or & dist\,(x, S \setminus \{x\}) \neq 0 \\ i.e, & x \in acc(S) & or & x \in iso(S)\,. \end{array}$$

Thus for any set S in any metric space,

$$S = iso\,(S) \cup (acc(S) \cap S)\,.$$

Also, from definition, $iso(S) \subseteq S$, but with the set of accumulation points we can have many possibilities. In the rest of this section we will explore all these possibilities.

Example 3.5.2 For any $x \in \mathbb{R}$,

$$dist\,(x, \mathbb{Q} \setminus \{x\}) = 0 = dist\,(x, \mathbb{I} \setminus \{x\})$$

where $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$.

Consider any positive real number ϵ. Since every interval contains infinitely many rational numbers, therefore $\mathbb{Q} \cap (x, x+\epsilon) \neq \emptyset$. Choose any $y \in \mathbb{Q} \cap (x, x+\epsilon)$, then

$$\begin{aligned} dist\ (x, \mathbb{Q} \setminus \{x\}) &= \inf\{d(x,a) \ : \ a \in \mathbb{Q} \setminus \{x\}\} \\ &\leq d(x,y) \\ &= |x-y| \\ &< \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get $dist(x, \mathbb{Q} \setminus \{x\}) = 0$.

Similarly we can show that for any x in $\mathbb{R}$, $dist(x, \mathbb{I} \setminus \{x\}) = 0 = dist\ (x, \mathbb{R} \setminus \{x\})$.

Hence it follows that

$$\boldsymbol{acc}\ (\mathbb{Q}) = \boldsymbol{acc}\ (\mathbb{I}) = \boldsymbol{acc}\ (\mathbb{R}) = \mathbb{R}\ . \ \blacksquare$$

From above example we see that two disjoint sets can have same set of accumulation points. Also, the set of accumulation points may properly contain the set, unlike isolated points.

Example 3.5.3 Consider the metric space $(\mathbb{R}, u)$, where u is the usual metric on $\mathbb{R}$. Then every point of $[0,1]$ is an accumulation point of the open interval $(0,1)$.

To prove this take any point x in $[0,1]$ and ϵ be any positive real number. Clearly,

$$(x-\epsilon, x+\epsilon) \cap (0,1)\setminus\{x\} \neq \emptyset\,.$$

Choose a point $y_\epsilon \in (x-\epsilon, x+\epsilon) \cap (0,1)\setminus\{x\}$, then obviously

$$dist(x, (0,1)\setminus\{x\}) \ \leq d(x, y_\epsilon) < \epsilon.$$

Since ϵ is arbitrary, we get

$$dist(x, (0,1)\setminus\{x\}) = 0\,.$$

Hence x is an accumulation point of $(0,1)$. Again as x is an arbitrary point of $[0,1]$, thus every point of $[0,1]$ is an accumulation point of $(0,1)$. $\blacksquare$

Example 3.5.4 Consider the real line $(\mathbb{R}, u)$ where u is the usual metric on $\mathbb{R}$. We will see that the set of integers, $\mathbb{Z}$, has no accumulation point.

Proof: Consider any $x \in \mathbb{R}$.

<u>Case 1.</u>$x$is an integer

Then we claim that

$$dist(x, \mathbb{Z}\setminus\{x\}) \geq 1/2\,.$$

Suppose on the contrary, $dist(x, \mathbb{Z}\setminus\{x\}) < 1/2$. By Theorem 2.5, there exists $y \in \mathbb{Z}\setminus\{x\}$ such that $d(x,y) < 1/2$. But then it implies that

$$y \in (x-1/2, x+1/2) \cap \mathbb{Z}\setminus\{x\}\,,$$

a contradiction to the fact that $(x-1/2, x+1/2) \cap \mathbb{Z}\setminus\{x\} = \emptyset$.

<u>Case 2.</u>$x$is any point in $\mathbb{R}\setminus\mathbb{Z}$

Then there exists an integer p such that $p < x < p+1$. Let

$$k = (1/2)\min\{x-p, p+1-x\}$$

Clearly, by the choice $(x-k, x+k) \cap \mathbb{Z} \setminus \{x\} = \emptyset$.

Claim:$dist(x,\mathbb{Z}\backslash\{x\}) \geq k$.

Suppose on the contrary,$dist(x,\mathbb{Z}\backslash\{x\}) < k$. There exists a point $y \in \mathbb{Z}\backslash\{x\}$ such that $d(x,y) < k$. But then$y \in (x-k, x+k) \cap \mathbb{Z}\backslash\{x\}$, a contradiction. Thus $dist(x,\mathbb{Z}\backslash\{x\}) \geq k$.

Thus in either case $dist(x,\mathbb{Z}\backslash\{x\}) > 0$, and hence x is not an accumulation point of $\mathbb{Z}$. Since $x \in \mathbb{R}$is an arbitrary real number, therefore no real number is an accumulation point of $\mathbb{Z}$. Thus $acc(\mathbb{Z}) = \emptyset$. ■

On the real line $(\mathbb{R}, u)$, the set of integers has no accumulation point but every integer is an accumulation point of $\mathbb{R}$ i.e., $acc\,(\mathbb{Z}) = \phi$ and $\mathbb{Z} \subseteq acc\,(\mathbb{R})$.

Example 3.5.5 On the real line $(\mathbb{R}, u)$ the set $S = \{1/n : n \in \mathbb{Z}^+\}$ has only one accumulation point, namely, zero.

Firstly we will show that 0 is an accumulation point of S. Let $\epsilon > 0$ be any real number. Then by Archimedean property, there exists an integer m such that $1/m < \epsilon$. Obviously, $u(0, 1/m) < \epsilon$ and hence $dist(0,S) < \epsilon$. Since ϵ is any positive real number, therefore it follows that $dist(0,S) = 0$. Thus 0 is an accumulation point of S.

Next we will show that no other real number can be an accumulation point of S. Let $x \neq 0$ be any real number. We have three possibilities:

Case 1.$x \in (-\infty,\ 0)$

Then for $k = |x|/2$,

$$\begin{aligned}
& (x-k, x+k) \cap S = \emptyset \\
\Rightarrow \quad & y \notin (x-k, x+k) \quad \forall\ y \in S \\
\Rightarrow \quad & |x-y| \geq k \quad \forall\ y \in S \\
\Rightarrow \quad & u(x,y) \geq k \quad \forall\ y \in S \\
\Rightarrow \quad & dist(x,S) = \inf_{y \in S} u(x,y) \geq k \\
\Rightarrow \quad & dist\,(x, S \setminus \{x\}) \geq k\ . \qquad [\because\ dist\,(x, S \setminus \{x\}) \geq dist(x,S)]
\end{aligned}$$

Case 2.$x \in (1,\ \infty)$

Then for $k = (x-1)/2$, $(x-k, x+k) \cap S = \emptyset$ and hence it follows that $dist\,(x, S \setminus \{x\}) \geq k$.

Case 3.$x \in (0,\ 1)$and$x \notin S$

Then there exists a positive integer m such that $m < 1/x < m+1$ $i.e.,$ $1/(m+1) < x < 1/m$. Setting $k = (1/2)\min\{x - (1/(m+1)),\ (1/m) - x\}$, we get $(x-k, x+k) \cap S = \emptyset$. Therefore $dist\,(x, S \setminus \{x\}) \geq k$.

Case 4.$x \in S$and$x \neq 1$

Then $x = 1/m$ for some positive integer m. Setting

$$k = (1/2)\min\{x - (1/(m+1)),\ (1/(m-1)) - x\}$$

we get $(x-k, x+k) \cap S \setminus \{x\} = \phi$ which further implies that $dist(x, S\backslash\{x\}) \geq k$.

Case 5.$x = 1$

Then setting $k = 1/4$, we get $(x-k, x+k) \cap S \setminus \{x\} = \phi$ and therefore $dist(x, S\backslash\{x\}) \geq k$.

Thus in all the cases we showed that $dist(x, S\backslash\{x\}) > 0$ and hence $x \notin acc(S)$. Since $x \neq 0$ is an arbitrary real number, therefore $acc(S) = \{0\}$. ■

Theorem 3.5.6 Suppose (X, d) is a metric space, $z \in X$ and S is a subset of X.

(i). If $z \notin S$, then $z \in acc(S)$ if, and only if, $dist(z, S) = 0$.

(ii). If $z \in S$, then $z \in acc(S)$ if, and only if, $z \notin iso(S)$.

(iii). $z \in acc(S)$if, and only if, $z \notin iso(S)$ and $dist(z, S) = 0$.

Proof: (i) If $z \notin S$, then

$$\begin{aligned} z \in acc(S) &\Leftrightarrow dist\,(z, S \setminus \{z\}) = 0 \\ &\Leftrightarrow dist\,(z, S) = 0 \qquad [\because S \setminus \{z\} = S] \end{aligned}$$

(ii) If $z \in S$, then

$$z \notin iso(S) \Leftrightarrow dist(z, S\backslash\{z\}) = 0 \Leftrightarrow z \in acc(S).$$

(iii) If $z \notin S$, then

$$\begin{aligned} z \in acc(S) &\Leftrightarrow dist\,(z, S \setminus \{z\}) = 0 \\ &\Leftrightarrow dist\,(z, S) = 0 \qquad [\because S \setminus \{z\} = S] \\ &\Leftrightarrow dist\,(z, S) = 0 \quad and \quad z \notin iso(S)[\because z \notin S \Rightarrow z \notin iso(S)]. \end{aligned}$$

If $z \in S$, then

$$\begin{aligned} z \in acc(S) &\Leftrightarrow dist\,(z, S \setminus \{z\}) = 0 \\ &\Leftrightarrow z \notin iso(S) \\ &\Leftrightarrow dist\,(z, S) = 0 \quad and \quad z \notin iso(S)[\because z \in S \Rightarrow dist\,(z, S) = 0] \end{aligned}$$

Thus (iii) follows. ■

Theorem 3.5.7 Suppose (X, d) is a metric space and $A \subseteq B \subseteq X$. Then $acc(A) \subseteq acc(B)$.

Proof: Let $x \in acc(A)$. Then $dist(x, A\backslash\{x\}) = 0$. Since $A\backslash\{x\} \subseteq B\backslash\{x\}$, therefore by Theorem 3.3.1

$$dist(x, B\backslash\{x\}) \leq dist(x, A\backslash\{x\}) = 0.$$

Hence it follows that $dist(x, B\backslash\{x\}) = 0$, which further implies that $x \in acc(B)$. Thus $acc(A) \subseteq acc(B)$.■

Theorem 3.5.8 Suppose (X, d) is a metric space and $A \subseteq B \subseteq X$. Then

$$acc_B(A) = B \cap acc_X(A).$$

Proof: Obviously, for any $x \in B$ we have $dist_B(x, A\backslash\{x\}) = dist_X(x, A\backslash\{x\})$.

Now consider

$$x \in acc_B(A)$$

$$\Leftrightarrow x \in B \text{ and } dist_B(x, A\backslash\{x\}) = 0$$

$$\Leftrightarrow x \in B \text{ and } dist_X(x, A\backslash\{x\}) = 0$$

$$\Leftrightarrow x \in B \cap acc_X(A)$$

Hence the theorem. ■

Example 3.5.9 Consider the real line $(\mathbb{R}, u)$ with u as its usual metric. Let

$$A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \quad and \quad B = \{n : n \in \mathbb{N}\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}.$$

Clearly, $A \subseteq B \subseteq \mathbb{R}$. Moreover, in view of Example 3.5.5, it is easy to see that

$$acc_B(A) = \emptyset \quad and \quad acc_{\mathbb{R}}(A) = \{0\}.$$

Consequently, $cc_B(A) = B \cap acc_{\mathbb{R}}(A)$. ■

4. SOLVED PROBLEMS

Problem 1. Let (X, d) be a discrete metric space and S be any subset of X. Show that

$$acc\,(S) = \emptyset \,.$$

Proof: Since

$$dist\,(x, S \setminus \{x\}) = \begin{cases} \infty & if \;\; S = \{x\} \;\; or \;\; S = \emptyset \\ \\ 1 & otherwise \end{cases} \qquad \forall\; x \in X.$$

It follows that $dist\,(x, S \setminus \{x\}) \neq 0$ for all $x \in X$. Hence we get $acc\,(S) = \emptyset$. ■

Problem 2. Suppose (X, d) is a metric space with more than one point. Find subsets A and B of X such that $diam(A \cup B) \geq diam(A) + diam(B)$.

Proof: Let $x, y \in X$ be two distincts points. Let $A = \{x\}$ and $B = \{y\}$. Then

$$diam(A) = 0 = diam(B) \;\; and \;\; diam(A \cup B) = d(x, y) > 0\,.$$

Thus $diam(A \cup B) \geq diam(A) + diam(B)$. ■

Problem 3. Find a condition on a metric space (X, d) that ensures that there exists subsets A and B of X with $A \subset B$ such that $diam(A) = diam(B)$.

Proof: If (X, d) is a metric space with $|X| \geq 3$ having three points x_1, x_2, x_3 such that

$$d(x_i, x_j) = c \qquad \forall\, i, j\; (i \neq j)$$

Where $c > 0$ is some real number. Then $A = \{x_1, x_2\}$ and $B = \{x_1, x_2, x_3\}$ are such that

$$A \subset B \quad and \quad diam(A) = diam(B). \;\; ■$$

Problem 4. Consider the metric subspace $X = I \cup J$ of $\mathbb{R}$, where I is the interval $(0, 1)$ and J is the interval $[4, 7)$. Show that $dist\,(sup_X\, I, I) \neq 0$.

Proof: Clearly, the set of all upper bounds of I in X is the set J where

$$J = \{x \in X : y \leq x \;\; for\, all\; y \in I\}.$$

Moreover, $\inf J = 4$, thus it follows that $sup_X\, I = 4$. Hence $dist(sup_X\, I, I) = 3 > 0$. ■

Problem 5. Find a metric space (X, d), an element x of X and non-empty subsets A and B of X with $A \subseteq B$ such that $dist\,(x, A) > dist\,(x, B) + diam\,(B \backslash A)$.

Proof: Let $X = \mathbb{R}$ and $d : X \times X \to \{0,1\}$ defined as

$$d(x, y) = \begin{cases} 1 & if \;\; x \neq y \\ 0 & if \;\; x = y\,. \end{cases}$$

For $x = 5$, $A = [1,2]$, $B = [1,2] \cup \{5\}$, we have $dist\,(x, A) = 1$, $dist\,(x, B) = 0$ and $diam\,(B \backslash A) = 0$. Thus

$$dist\,(x, A) > dist\,(x, B) + diam\,(B \backslash A). \;\; ■$$

Note that we can also consider real line $(\mathbb{R}, u)$, then for $x = 5$, $A = [1,2]$, $B = [1,2] \cup \{5\}$ we will have

$$dist\,(x, A) > dist\,(x, B) + diam\,(B \backslash A).$$

Problem 6. Suppose (X, d) is a metric space and that S is a subset of X. Show that $iso(S) = S \setminus acc(S)$.

Proof: Consider

$$\begin{aligned} x \in iso(S) \quad &\Leftrightarrow \quad x \in S \quad and \;\; dist(x, S \setminus \{x\}) > 0 \\ &\Leftrightarrow \quad x \in S \quad and \;\; x \notin acc(S) \\ &\Leftrightarrow \quad x \notin S \setminus acc(S) \end{aligned}$$

Hence it follows that $iso(S) = S \setminus acc(S)$. ■

Problem 7. Let (X, d) be a metric space and $F \subseteq X$ be a finite set. Show that $acc(F) = \emptyset$.

Proof: If $F = \phi$, then $dist\,(x, F \setminus \{x\}) = \infty \;\; \forall \; x \in X$. Hence $acc\,(F) = \phi$. Let $F = \{p_1, p_2, \dots \dots, p_k\}$ and x be any point in X.

Case 1. $x \notin F$

Then for each $i \; (1 \le i \le k)$, $x \ne p_i$ and hence we must have $d(x, p_i) > 0$. Therefore

$$dist(x, F \backslash \{x\}) = dist(x, F) = \min\{d(x, p_i) \colon \; 1 \le i \le k\} > 0.$$

It follows that $x \notin acc(F)$.

Case 2. $x \in F$

Then $x = p_j$ for some j, $(1 \le j \le k)$. Clearly, for any $i(\ne j)$, $d(x, p_i) > 0$. Hence

$$dist(x, F \backslash \{x\}) = \min\{d(x, p_i) \colon \; 1 \le i \le k, i \ne j\} > 0$$

and therefore $x \notin acc(F)$.

Thus in either case $x \notin acc(F)$. Since $x \in X$ is arbitrary, it follows that $acc(F) = \emptyset$. ■

Problem 8. Let $\mathcal{C}$ be a non-empty collection of subsets of a metric space (X, d) and $S \in \mathcal{C}$. Prove that every isolated point of S that belongs to $\bigcap \mathcal{C}$ is an isolated point of $\bigcap \mathcal{C}$ i.e.,

$$iso(S) \cap (\bigcap \mathcal{C}) \subseteq iso(\bigcap \mathcal{C}).$$

Proof: Since $\bigcap\mathcal{C} \subseteq S \subseteq X$, so from Theorem 3.4.7 $iso(S) \cap (\bigcap\mathcal{C}) \subseteq iso(\bigcap\mathcal{C})$. ■

Problem 9. Let $\mathcal{C}$ be a non-empty collection of subsets of a metric space (X, d). Prove that every isolated point of $\bigcup\mathcal{C}$ is isolated in every member of $\mathcal{C}$ to which it belongs i.e.,

$$x \in iso\,(\bigcup \mathcal{C}) \Rightarrow x \in \bigcap_{S \in \mathcal{C}_x} iso\,(S)$$

where $\mathcal{C}_x = \{S \in \mathcal{C} : x \in S\}$ is the set of all sets in $\mathcal{C}$ that contain x.

Proof: Since $S \subseteq \bigcup\mathcal{C} \subseteq X$ for any $S \in \mathcal{C}_x$, therefore from Theorem 3.4.7

$$iso(\bigcup\mathcal{C}) \cap S \subseteq iso\,(S).$$

Further since $x \in iso(\bigcup\mathcal{C}) \cap S$, it follows that $x \in S$. Again as $S \in \mathcal{C}_x$ is an arbitrary element, thus

$$x \in \bigcap_{S \in \mathcal{C}_x} iso\,(S) \;. \;\; ■$$

Problem 10. Let $\mathcal{C}$ be a non-empty collection of subsets of a metric space(X,d). Prove that

a) $acc\,(\cap\,\mathcal{C}) \subseteq \cap\{acc\,(S) : S \in\, \mathcal{C}\}$

b) $\cup\{acc\,(S) : S \in\, \mathcal{C}\} \subseteq acc\,(\cup\,\mathcal{C})$.

Moreover, show that both the inclusions may be proper.

Proof:a) Let $x \in acc\,(\cap\,\mathcal{C})$ be any point and $S \in \mathcal{C}$ be any arbitrary set. Then

$$\cap\mathcal{C} \setminus \{x\} \subseteq S \setminus \{x\}$$
$$\Rightarrow\ dist\,(x, S \setminus \{x\}) \leq dist\,(x, \cap\mathcal{C} \setminus \{x\}) = 0 \qquad [\because x \in acc(\cap\mathcal{C})]$$
$$\Rightarrow\ dist\,(x, S \setminus \{x\}) = 0$$
$$\Rightarrow\ x \in acc\,(S).$$

Since $x \in acc\,(\cap\,\mathcal{C})$ and $S \in \mathcal{C}$ are arbitrary, it follows that

$$acc\,(\cap\mathcal{C}) \subseteq \cap\{acc\,(S) : S \in\, \mathcal{C}\}.$$

Inclusion may be proper. For example consider the real line $(\mathbb{R}, u)$. Let

$$\mathcal{C} = \{(0,x) :\ x \in \mathbb{R}^+\}.$$

Then $\cap\mathcal{C} = \phi$. Thus $acc\,(\cap\mathcal{C}) = \phi$. Also, for each $x \in \mathbb{R}^+$,

$$acc\,\big((0,x)\big) = [0,x]$$
$$\Rightarrow \quad \bigcap_{x\in\mathbb{R}^+} acc\,\big((0,x)\big) = \bigcap_{x\in\mathbb{R}^+} [0,x] = \{0\}.$$

Hence the inclusion in this case is proper.

b)Let $S \in \mathcal{C}$ be any arbitrary set. Also, let $x \in acc\,(S)$ be any element. Now

$$S\setminus\{x\} \subseteq \cup\mathcal{C} \setminus \{x\}$$
$$\Rightarrow\ dist\,(x, \cup\mathcal{C} \setminus \{x\}) \leq dist\,(x, S \setminus \{x\}) = 0 \qquad [\because x \in acc(S)]$$
$$\Rightarrow\ dist\,(x, \cup\mathcal{C} \setminus \{x\}) = 0$$
$$\Rightarrow\ x \in acc\,(\cup\mathcal{C}).$$

Thus $acc\,(S) \subseteq acc\,(\cup\mathcal{C})$ and consequently, $\cup\{acc\,(S) : S \in\, \mathcal{C}\} \subseteq acc\,(\cup\,\mathcal{C})$.

Again in this case inclusions may be proper. For example consider the real line $(\mathbb{R}, u)$. Let

$$\mathcal{C} = \{(x,\infty) :\ x \in \mathbb{R}^+\}.$$

Then $\cup\mathcal{C} = (0,\infty)$. Thus $acc\,(\cup\mathcal{C}) = [0,\infty)$. Also, for each $x \in \mathbb{R}^+$,

$$acc\,\big((x,\infty)\big) = [x,\infty)$$
$$\Rightarrow \quad \bigcup_{x\in\mathbb{R}^+} acc\,\big((x,\infty)\big) = \bigcup_{x\in\mathbb{R}^+} [x,\infty) = (0,\infty)$$
$$\Rightarrow \quad \bigcup_{x\in\mathbb{R}^+} acc\,\big((x,\infty)\big) \subset acc\,(\cup\mathcal{C})$$

Hence the inclusion in this case is proper. ∎

Problem 11. Let (X,d) be a metric space and S be a subset of X. Show that if $p \notin acc(S)$, then there exists a real $r > 0$ such that $dist(p, acc(S)) \geq r$.

Proof: Since $p \notin acc(S)$,$dist\,(p, S\backslash\{p\}) > 0$ and there exists a real number $r > 0$ such that

$$dist(p, S\backslash\{p\}) > r.$$

We claim that for any $q \in acc(S)$, $d(p,q) \geq r$.

Consider any $q \in acc(S)$, then clearly by definition $dist(q, S\backslash\{q\}) = 0$. Therefore for any real number ϵ $(0 < \epsilon < r)$, there exists a point x in S such that $0 < d(q,x) < \epsilon/2$. We will show that $x \neq p$. Suppose on the contrary, $x = p$, then $d(q,p) < \epsilon/2$. Let ϵ_1 be any real number such that $0 < \epsilon_1 < d(q,p)$. Since $dist(q, S\backslash\{q\}) = 0$, there exists $y \in S$ such that $d(q,y) < \epsilon_1/2$.

Clearly, $y \neq p$. Therefore as $dist(p, S\backslash\{p\}) > r$, we have $d(p,y) > r$. Now

$$\begin{aligned} r < d(p,y) &\leq d(q,p) + d(q,y) \\ &< \epsilon/2 + \epsilon_1/2 \\ &< r/2 + r/2 = r, \end{aligned}$$

a contradiction. Thus our assumption is wrong and $x \neq p$. Then $d(p,x) > r$. Again consider

$$\begin{aligned} r < d(p,x) &\leq d(q,p) + d(q,x) \\ &< d(q,p) + \epsilon/2\,. \end{aligned}$$

Hence it follows that $d(p,q) > r - \epsilon/2$. Since $\epsilon(< r)$ is arbitrary, therefore letting $\epsilon \to 0$ we get $d(p,q) \geq r$. Again since $q \in acc(S)$ is arbitrary, therefore $dist(p, acc(S)) \geq r$. ■

Problem 12. Suppose$n \in \mathbb{N}$ and, for each$i \in \mathbb{N}_n$, (X_i, τ_i) is a metric space. Let $d$be a conserving metric on$P = \prod_{i=1}^{n} X_i$. Suppose$S \subseteq P$and $a \in S$. Is it true that

$$a \in iso\,(S) \quad \Leftrightarrow \quad a_i \in iso\,(\pi_i(S)) \quad \forall\, i \in \mathbb{N}_n\,,$$

Whereπ_i denotes the natural projection of P onto X_i?

Proof: The implication $a \in iso\,(S)$ whenever $a_i \in iso\,(\pi_i(S))$ $\forall\, i \in \mathbb{N}_n$ is true in general. In fact since $a_i \in iso\,(\pi_i(S))$ $\forall\, i \in \mathbb{N}_n$,

$$\begin{aligned} & dist\,(a_i, \pi_i(S)\backslash\{a_i\}) > 0 \;\; \forall\;\; i \in \mathbb{N}_n \\ \Rightarrow\;\; & k = \min\{\, dist\,(a_i, \pi_i(S)\backslash\{a_i\}) \;:\; i \in \mathbb{N}_n\} > 0\,. \end{aligned}$$

We claim that $dist\,(a, S \setminus \{a\}) > k$. Let $\epsilon > 0$ be any real number. Then by definition of $dist\,(a, S \setminus \{a\})$, there exists $x \in S\backslash\{a\}$ such that

$$d(a,x) \leq dist\,(a, S \setminus \{a\}) + \epsilon\,.$$

As $x \neq a$, there exists $j \in \mathbb{N}_n$ such that $x_j \neq a_j$. Now since$d$ is conserving metric,

$$\begin{aligned} d(a,x) &\geq \max\{\tau_i(a_i, x_i) \;:\; i \in \mathbb{N}_n\} \\ &\geq \tau_j(a_j, x_j) \\ &\geq dist\,(a_j, \pi_j(S)\backslash\{a_j\}) > k\,. \end{aligned}$$

Thus $d(a,x) > k$ and consequently, $k < dist\,(a, S\backslash\{a\}) + \epsilon$. Since ϵ is arbitrary, we get

$$\begin{aligned} & dist\,(a, S\backslash\{a\}) \geq k > 0 \\ \Rightarrow\;\; & a \in iso\,(S). \end{aligned}$$

Thus $a \in iso\,(S)$whenever $a_i \in iso\left(\pi_i(S)\right)\ \ \forall\, i \in \mathbb{N}_n$.

The converse is true in case $S = P$ i.e. if $S = P$, then

$$a \in iso\,(S) \Rightarrow\ a_i \in iso\left(\pi_i(S)\right) \forall\, i \in \mathbb{N}_n\,.$$

Consider any $i \in \mathbb{N}_n$. We claim that $a_i \in iso\left(\pi_i(S)\right)$. Let $\epsilon > 0$ be any real number. Then by definition of $dist\,(a_i, \pi_i(S)\backslash\{a_i\})$, there exists $w \in \pi_i(S)\backslash\{a_i\}$ such that

$$dist\,(a_i, \pi_i(S)\backslash\{a_i\}) + \epsilon > \tau_i(a_i, w)\,.$$

Let $x \in S$ be such that

$$x_j = \begin{cases} a_j & if\ j \neq i \\ w & if\ j = i \end{cases}.$$

(Note that since $S = P$, we can choose such a $x \in S$)

Now consider,

$$\begin{aligned} dist\,(a, S\backslash\{a\}) &\leq\ d(a, x) \\ &\leq \sum_{j=1}^{n} \tau_j\left(a_j, x_j\right) [\because d\ \ is\ conserving\ metric] \\ &= \tau_i(a_i, w)\left[\because \tau_j\left(a_j, x_j\right) = \tau_j\left(a_j, a_j\right) = 0\ \ \forall j \neq i\,\right] \\ &\leq dist\,(a_i, \pi_i(S)\backslash\{a_i\}) + \epsilon\,. \end{aligned}$$

Thus $dist\,(a, S\backslash\{a\}) \leq dist\,(a_i, \pi_i(S)\backslash\{a_i\}) + \epsilon$. Letting $\epsilon \to 0$, we get

$$dist\,(a, S\backslash\{a\}) \leq dist\,(a_i, \pi_i(S)\backslash\{a_i\})\,.$$

Since $i \in \mathbb{N}_n$ is arbitrary, therefore

$$\begin{aligned} & dist\,(a, S\backslash\{a\}) \leq dist\,(a_i, \pi_i(S)\backslash\{a_i\}) && \forall\ i \in \mathbb{N}_n \\ \Rightarrow\ \ & a \in iso\,(S) \Rightarrow\ a_i \in iso\left(\pi_i(S)\right) && \forall\ i \in \mathbb{N}_n\,. \end{aligned}$$

Hence the claim.

If $S \neq P$, then converse may or may not hold. For example consider the Euclidean space $\mathbb{R}^2$ and the subset $S$of $\mathbb{R}^2$, where

$$\begin{aligned} S &= \{(a, b)\ : a \in \mathbb{R}^+\ and\ b = 1\ \ \ or\ \ a = 1\ \ and\ b \in \mathbb{R}^+\} \cup \{(0,0)\} \\ &= (\mathbb{R}^+ \times \{1\}) \cup (\{1\} \times \mathbb{R}^+) \cup \{(0,0)\}\,. \end{aligned}$$

Then clearly, $\pi_1(S) = \mathbb{R}^+ \cup \{0\} = \pi_2(S)$. Clearly, 0 is not an isolated point of both $\pi_1(S)$and $\pi_2(S)$. But since $dist\left((0,0),\ S \setminus \{(0,0)\}\right) = 1$, therefore $(0,0)$ is an isolated point of S. ■

Problem 13. Suppose$n \in \mathbb{N}$ and, for each$i \in \mathbb{N}_n$, (X_i, τ_i) is a metric space. Let $d$be a conserving metric on$P = \prod_{i=1}^{n} X_i$. Suppose$S \subseteq P$and $a \in S$. Is it true that

$$a \in acc\,(S)\quad \Leftrightarrow\quad a_j\ \in acc\ \left(\pi_j\,(S)\right)\ \ for\ some\ j \in \mathbb{N}_n\,,$$

Whereπ_j denotes the natural projection of P onto X_j?

Proof: Again as in Problem 11, there is implication in one direction. The implication

$$a \in acc\,(S)\ \ \Rightarrow\ a_j\ \in acc\ \left(\pi_j\,(S)\right)\ \ for\ some\ j \in \mathbb{N}_n$$

is true in general.Let $\epsilon > 0$ be any real number. Then by definition of $dist\,(a, S \setminus \{a\})$, there exists $x \in S \setminus \{a\}$ such that

$$d(a,x) \leq dist\,(a, S \setminus \{a\}) + \epsilon = \epsilon. \qquad [\because a \in acc\,(S) \Rightarrow dist\,(a, S \setminus \{a\})]$$

As $x \neq a$, there exists $j \in \mathbb{N}_n$ such that $x_j \neq a_j$. Now

$$\begin{aligned} \epsilon &\geq d(a,x) \\ &\geq \max\{\tau_i(a_i, x_i) \;:\; i \in \mathbb{N}_n\} [\because d \;\; is\ conserving\ metric] \\ &\geq \tau_j(a_j, x_j) \\ &\geq dist\,(a_j, \pi_j(S) \setminus \{a_j\}). \end{aligned}$$

Thus $dist\,(a_j, \pi_j(S) \setminus \{a_j\}) \leq \epsilon$. Letting $\epsilon \to 0$, we get

$$\begin{aligned} & dist\,(a_j, \pi_j(S) \setminus \{a_j\}) = 0 \\ \Rightarrow\; & a_j \in acc\,(\pi_j(S)) \end{aligned}$$

Hence $a_j \in acc\,(\pi_j(S))$ $for\ some\ j \in \mathbb{N}_n$ whenever $a \in acc\,(S)$.

The converse is true in case $S = P$ i.e. if $S = P$, then

$$a_j \in acc\,(\pi_j(S)) \;\; for\ some\ j \in \mathbb{N}_n \;\; \Rightarrow \;\; a \in iso\,(S)\ .$$

Let $\epsilon > 0$ be any real number. Then by definition of $dist\,(a_j, \pi_j(S) \setminus \{a_j\})$, there exists $w \in \pi_j(S) \setminus \{a_j\}$ such that

$$dist\,(a_j, \pi_j(S) \setminus \{a_j\}) + \epsilon > \tau_j(a_j, w).$$

Let $x \in S$ be such that

$$x_i = \begin{cases} a_i & if\ i \neq j \\ w & if\ i = j \end{cases}.$$

(Note that since $S = P$, we can choose such a $x \in S$)

Now consider,

$$\begin{aligned} dist\,(a, S \setminus \{a\}) &\leq d(a,x) \\ &\leq \sum_{i=1}^{n} \tau_i(a_i, x_i) [\because d \;\; is\ conserving\ metric] \\ &= \tau_j(a_j, w) [\because \tau_i(a_i, x_i) = \tau_i(a_i, a_i) = 0 \;\; \forall\, i \neq j\,] \\ &\leq dist\,(a_j, \pi_j(S) \setminus \{a_j\}) + \epsilon. \end{aligned}$$

Thus $dist\,(a, S \setminus \{a\}) \leq dist\,(a_j, \pi_j(S) \setminus \{a_j\}) + \epsilon$. Letting $\epsilon \to 0$, we get

$$dist\,(a, S \setminus \{a\}) \leq dist\,(a_j, \pi_j(S) \setminus \{a_j\}).$$

Since $a_j \in acc\,(\pi_j(S))$, $dist\,(a_j, \pi_j(S) \setminus \{a_j\}) = 0$ and therefore

$$\begin{aligned} & dist\,(a, S \setminus \{a\}) = 0 \\ \Rightarrow\; & a \in iso\,(S) \end{aligned}$$

Hence the claim.

If $S \neq P$, then converse may or may not hold. For example consider the Euclidean space $\mathbb{R}^2$ and the subset $S$of $\mathbb{R}^2$, where

$$S = \{(a,b)\ : a \in \mathbb{R}^+ \ and\ b = 1 \quad or\ \ a = 1\ \ and\ b \in \mathbb{R}^+\}$$
$$= (\mathbb{R}^+ \times \{1\}) \cup (\{1\} \times \mathbb{R}^+)\,.$$

Then $\pi_1(S) = \mathbb{R}^+ = \pi_2(S)$. Clearly, 0 is an accumulation point of both $\pi_1(S)$and $\pi_2(S)$. But since $dist\left((0,0),\ S \setminus \{(0,0)\}\right) = 1$, therefore $(0,0)$ is not an accumulation point of S. ∎

5. SUMMARY

In this chapter, we introduced the concept of diameter of any given set in any arbitrary metric space and discussed how the definition can be seen as a generalization of the concept of diameter of a circle. We observed that the diameter as a function preserves the order i.e., if $A \subseteq B$, then $diam(A) \leq diam(B)$. Further we proved that for any set $S \subseteq \mathbb{R}$, $diam(S) = \sup\, S - \inf\, S$ and hence enabling us to derive diameter on the real line easily. Following diameter we introduced the concept of distance of a point from a given set and discussed various interesting inequalities arising from this. Also, we proved that distance of any point on the real line is zero both from the set $\mathbb{Q}$ and the set $\mathbb{R}\setminus\mathbb{Q}$ of irrationals. Then we introduced the concepts of isolated points and accumulation points and discussed various examples to get familiarity with these concepts. We also discussed various interesting and important consequences arising from these concepts.

6. EXERCISES

Question 1. Let (X,d) be a metric space and S be a subset of X. Show that $p\ \in acc(S)$ if and only if for each $n\ \in\ \mathbb{Z}^+$, there exists a point $x_n\ \in S$ such that

$$dist(x_n, p) < \frac{1}{n}.$$

Question 2. Let (X,d) be a metric space and S be a subset of X. Show that $p\ \in acc(S)$ if and only if for each $k\ \in\ \mathbb{R}^+$, there exist infinitely many points of S at distance at most k from p i.e., the set $\{y\ \in S : d(p,y) \leq k\}$ is an infinite set. Further, deduce that on the real line $\mathbb{R}$, the set of integers $\mathbb{Z}$ has no accumulation point.

Question 3. Show that in the complex plane $\mathbb{C}$, the set of accumulation points of the disc $\{z\ \in\ \mathbb{C} :\ |z| \leq 1\}$ is the disc itself.

Question 4. Let (X,d) be a metric space and S be a subset of X. If a,b are any points in S, then

$$|d(a,S) - d(b,S)| \leq d(a,b).$$

7. REFRENCES

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